

On resolvent matrix, Dyukarev-Stieltjes parameters and orthogonal matrix polynomials via $[0, \infty)$ -Stieltjes transformed sequences

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By using Schur transformed sequences and Dyukarev-Stieltjes parameters we obtain a new representation of the resolvent matrix corresponding to the truncated matricial Stieltjes moment problem. Explicit relations between orthogonal matrix polynomials and matrix polynomials of the second kind constructed from consecutive Schur transformed sequences are obtained. Additionally, a non-negative Hermitian measure for which the matrix polynomials of the second kind are the orthogonal matrix polynomials is found.

Keywords:

Resolvent matrix, orthogonal matrix polynomials, Dyukarev-Stieltjes parameters, Schur transformed sequences.

1. Introduction

This paper is a continuation of work done in the papers [12, 13, 20, 25, 26], where two truncated matricial power moment problems on semi-infinite intervals made up one of the main topics. The starting point of studying such problems was the famous two part memoir of Stieltjes [37, 38] where the author's investigations on questions for special continued fractions led him to the power moment problem on the interval $[0, \infty)$. A complete theory of the treatment of power moment problems on semi-infinite intervals in the scalar case was developed by M. G. Kreĭn in collaboration with A. A. Nudelman (see [33, Section 10], [34], [35, Chapter V]). For a modern operator-theoretical treatment of the power moment problems named after Hamburger and Stieltjes and its interrelations, we refer the reader to Simon [36].

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1. Introduction

The matrix version of the classical Stieltjes moment problem was studied in Adamyan/Tkachenko [1, 2], Andô [4], Bolotnikov [5–7], Bolotnikov/Sakhnovich [8], Chen/Hu [9], Chen/Li [10], Dyukarev [16, 17], Dyukarev/Katsnel'son [22, 23], Hu/Chen [32].

The central research object of the present work is the resolvent matrix (RM) U_m of the truncated matricial Stieltjes matrix moment (TSMM) problem. The importance of the knowledge of the RM U_m is explained by the fact that the matrix U_m generates the solution set of the TSMM problem via a linear fractional transformation. The multiplicative decomposition of U_m in simplest factors containing Dyukarev-Stieltjes (DS) parameters \mathbf{M}_j and \mathbf{L}_j [12] allowed us to attain interesting interrelations between the orthogonal polynomials $P_{k,j}$, their second kind polynomials $Q_{k,j}$ and the DS-parameters \mathbf{M}_j and \mathbf{L}_j , as well as the Schur complements $\hat{H}_{k,j} = L_{k,j}$; see [12].

In the present work, inspired by Dyukarev's multiplicative decomposition of U_m (see Proposition 4.4), the Schur transformed sequences [28, 29], and the representation of the RM in terms of the polynomials $P_{k,j}$ and $Q_{k,j}$ [12], a factorization of the RM U_m is obtained. This representation is constructed through a sequence of ℓ -th Schur transformed sequences and corresponding polynomials $P_{k,j}^{(\ell)}$ and $Q_{k,j}^{(\ell)}$. An important consequence of such representation is the fact that a non-negative Hermitian measure for which the polynomials of the second kind $Q_{k,j}^{(1)}$ are orthogonal is explained. By employing the interrelations between the orthogonal matrix polynomials and the Hurwitz type matrix polynomials (see [13]) new identities involving $P_{k,j}$, $Q_{k,j}$ and $P_{k,j}^{(1)}$, $Q_{k,j}^{(1)}$ are attained; see Theorem 8.8. The scalar version of the mentioned interrelations were studied in [14].

The starting point of our considerations in this paper is the Dyukarev-Stieltjes parametrization $[(\mathbf{L}_k)_{k=0}^\infty, (\mathbf{M}_k)_{k=0}^\infty]$ of a matricial moment sequence corresponding to a completely non-degenerate non-negative Hermitian measure on the right half-axis $[0, \infty)$, having moments up to any order. Yu. M. Dyukarev [19] introduced these parameters in connection with a multiplicative decomposition of his resolvent matrix for the truncated matricial Stieltjes moment problem into elementary factors to characterize indeterminacy. In the scalar case the Dyukarev-Stieltjes parametrization coincides with the classical parameters $[(l_k)_{k=0}^\infty, (m_k)_{k=0}^\infty]$ used by Stieltjes [37, 38] to formulate his indeterminacy criterion. M. G. Kreĭn gave a mechanical interpretation for Stieltjes' investigations on continued fractions (see Gantmacher/Kreĭn [31, Anhang 2] or Akhiezer [3, Appendix]) by a weightless thread carrying point masses m_k with intermediate distances l_k . In [25] another parametrization of moment sequences related to a semi-infinite interval $[\alpha, \infty)$ was introduced, the so-called Stieltjes parametrization $(\mathfrak{k}_j)_{j=0}^\infty$. This parametrization is strongly connected to a Schur-type algorithm considered in [9, 28, 29, 32] for solving the truncated matricial Stieltjes moment problem step-by-step by reducing the number of given data. In one step the sequence of prescribed moments is transformed into a shorter sequence, the so-called Schur transform, eliminating one moment. This procedure is equivalent to dropping the first Stieltjes parameter \mathfrak{k}_0 . In Theorem 6.6 we show that this transformation is also essentially equivalent to dropping the Dyukarev-Stieltjes parameter \mathbf{M}_0 and interchanging the roles of \mathbf{L}_k and \mathbf{M}_k , which is especially interesting against the background of M. G. Kreĭn's mechanical interpretation. By dividing

1. Introduction

the elementary factors of Yu. M. Dyukarev's above mentioned multiplicative representation of his resolvent matrix into two groups, in Theorem 7.12 we give a factorization of this resolvent matrix, which corresponds to a splitting up of the original problem into two smaller moment problems associated with the first part of the original sequence of prescribed moments and a second part obtained by repeated application of the Schur transformation. Comparing blocks in this formula, in Theorem 8.3 we can represent the orthogonal matrix polynomials and second kind matrix polynomials with respect to the Schur transformed moment sequence in terms of the polynomials corresponding to the original moment sequence. In particular, we state orthogonality relations for the matrix polynomials of the second kind in Proposition 8.4.

In order to formulate the moment problems we are going to study, we first review some notation. Let \mathbb{C} , \mathbb{R} , \mathbb{N}_0 , and \mathbb{N} be the set of all complex numbers, the set of all real numbers, the set of all non-negative integers, and the set of all positive integers, respectively. Throughout this paper, let $p, q \in \mathbb{N}$. For all $\alpha, \beta \in \mathbb{R} \cup \{-\infty, \infty\}$, let $\mathbb{Z}_{\alpha, \beta}$ be the set of all integers k for which $\alpha \leq k \leq \beta$ holds. If \mathcal{X} is a nonempty set, then $\mathcal{X}^{p \times q}$ stands for the set of all $p \times q$ matrices, each entry of which belongs to \mathcal{X} , and \mathcal{X}^p is short for $\mathcal{X}^{p \times 1}$. If (Ω, \mathfrak{A}) is a measurable space, then each countably additive mapping whose domain is \mathfrak{A} and whose values belong to the set $\mathbb{C}_{\geq}^{q \times q}$ of all non-negative Hermitian complex $q \times q$ matrices is called a non-negative Hermitian $q \times q$ measure on (Ω, \mathfrak{A}) . Denote by $\mathbb{C}_{>}^{q \times q}$ the set of all positive Hermitian complex $q \times q$ matrices.

Let $\mathfrak{B}_{[0, \infty)}$ be the σ -algebra of all Borel subsets of $[0, \infty)$, let $\mathcal{M}_{\geq}^q([0, \infty))$ be the set of all non-negative Hermitian $q \times q$ measures on $([0, \infty), \mathfrak{B}_{[0, \infty)})$ and, for all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $\mathcal{M}_{\geq, \kappa}^q([0, \infty))$ be the set of all $\sigma \in \mathcal{M}_{\geq}^q([0, \infty))$ such that the integral

$$s_j^{(\sigma)} := \int_{[0, \infty)} t^j \sigma(dt) \quad (1.1)$$

exists for all $j \in \mathbb{Z}_{0, \kappa}$. Two matricial power moment problems lie in the background of our considerations. The first one is the following:

M $[[0, \infty); (s_j)_{j=0}^{\kappa}, =]$ Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{M}_{\geq}^q([0, \infty); (s_j)_{j=0}^{\kappa}, =]$ of all $\sigma \in \mathcal{M}_{\geq, \kappa}^q([0, \infty))$ for which $s_j^{(\sigma)} = s_j$ is fulfilled for all $j \in \mathbb{Z}_{0, \kappa}$.

The second matricial moment problem under consideration is a truncated one with an additional inequality condition for the last prescribed moment:

M $[[0, \infty); (s_j)_{j=0}^m, \leq]$ Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{M}_{\geq}^q([0, \infty); (s_j)_{j=0}^m, \leq]$ of all $\sigma \in \mathcal{M}_{\geq, m}^q([0, \infty))$ for which $s_m - s_m^{(\sigma)}$ is non-negative Hermitian and, in the case $m > 0$, moreover $s_j^{(\sigma)} = s_j$ is fulfilled for all $j \in \mathbb{Z}_{0, m-1}$.

In order to give a better motivation for our considerations in this paper, we are going to recall the characterizations of solvability of the above mentioned moment problems, which were obtained in [20]. This requires some preparations.

1. Introduction

For all $n \in \mathbb{N}_0$, let $\mathcal{K}_{q,2n+1}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices, such that the block Hankel matrices

$$H_{1,n} := [s_{j+k}]_{j,k=0}^n \quad \text{and} \quad H_{2,n} := [s_{j+k+1}]_{j,k=0}^n \quad (1.2)$$

are both non-negative Hermitian. For all $n \in \mathbb{N}_0$, let $\mathcal{K}_{q,2n}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices, such that $H_{1,n}$ is non-negative Hermitian and, in the case $n \geq 1$, furthermore $H_{2,n-1}$ is non-negative Hermitian. For all $m \in \mathbb{N}_0$, let $\mathcal{K}_{q,m}^{\geq,e}$ be the set of all sequences $(s_j)_{j=0}^m$ of complex $q \times q$ matrices for which a complex $q \times q$ matrix s_{m+1} exists such that $(s_j)_{j=0}^{m+1}$ belongs to $\mathcal{K}_{q,m+1}^{\geq}$. Let $\mathcal{K}_{q,\infty}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^m$ belongs to $\mathcal{K}_{q,m}^{\geq}$ for all $m \in \mathbb{N}_0$, and let $\mathcal{K}_{q,\infty}^{\geq,e} := \mathcal{K}_{q,\infty}^{\geq}$. For all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we call a sequence $(s_j)_{j=0}^{\kappa}$ *Stieltjes non-negative definite* (resp. *Stieltjes non-negative definite extendable*) if it belongs to $\mathcal{K}_{q,\kappa}^{\geq}$ (resp. to $\mathcal{K}_{q,\kappa}^{\geq,e}$). Observe that these notions coincide with the right-sided version in [25, Definition 1.3, p. 213] for $\alpha = 0$.

Using the sets of matrix sequences above, we are able to formulate the solvability criterions of the problems $\mathbf{M}[[0, \infty); (s_j)_{j=0}^m, =]$ and $\mathbf{M}[[0, \infty); (s_j)_{j=0}^m, \leq]$, which were obtained in [20] for intervals $[\alpha, \infty)$ with arbitrary $\alpha \in \mathbb{R}$:

Theorem 1.1. *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}_{\geq}^q[[0, \infty); (s_j)_{j=0}^{\kappa}, =] \neq \emptyset$ if and only if $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa}^{\geq,e}$.*

In the case $\kappa \in \mathbb{N}_0$, Theorem 1.1 is a special case of [20, Theorem 1.3, p. 909]. If $\kappa = \infty$, the asserted equivalence can be proved using the equation $\mathcal{M}_{\geq}^q[[0, \infty); (s_j)_{j=0}^m, =] = \bigcap_{m=0}^{\infty} \mathcal{M}_{\geq}^q[[0, \infty); (s_j)_{j=0}^m, =]$ and the matricial version of the Helly-Prohorov theorem (see [24, Satz 9, p. 388]). We omit the details of the proof, the essential idea of which is originated in [3, proof of Theorem 2.1.1, p. 30].

Theorem 1.2 (see [20, Theorem 1.4, p. 909]). *Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}_{\geq}^q[[0, \infty); (s_j)_{j=0}^m, \leq] \neq \emptyset$ if and only if $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^{\geq}$.*

The importance of Theorems 1.1 and 1.2 led us in [20, 25, 26] to a closer look at the properties of sequences of complex $q \times q$ matrices, which are Stieltjes non-negative definite or Stieltjes non-negative definite extendable. Guided by our former investigations on Hankel non-negative definite sequences and Hankel non-negative definite extendable sequences, which were done in [21], in [20, Section 4] and in [25] we started a thorough study of the structure of Stieltjes non-negative definite sequences and Stieltjes non-negative definite extendable sequences.

This paper is organized as follows: In Section 2 we recall some results on the Schur complement $H_{1,j}/H_{1,j-1}$ called Stieltjes parametrization $(\mathfrak{t}_j)_{j=0}^{\kappa}$ of a sequence $(s_j)_{j=0}^{\kappa}$. In Section 3 the Dyukarev-Stieltjes parametrization given by the sequence $[(\mathbf{L}_k)_{k=0}^{\infty}, (\mathbf{M}_k)_{k=0}^{\infty}]$ is recalled. Interrelations between this pair of sequences and the Stieltjes parametrization and vice versa are discussed. In Section 4, via a set of nonnegative column pairs $\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}$, the parametrization of the solution set in the non-degenerate case is rewritten in terms of

2. Stieltjes parametrization

a linear fractional transformation. Moreover, the representation of the resolvent matrix corresponding to a matricial truncated Stieltjes moment problem with matrix polynomials orthogonal on $[0, \infty)$ and matrix polynomials of the second kind is recalled. In Section 5 we consider the notion and main results concerning the Schur transform. Section 6 is devoted to the Dyukarev-Stieltjes parametrization $[(\mathbf{L}_k^{(\ell)})_{k=0}^\infty, (\mathbf{M}_k^{(\ell)})_{k=0}^\infty]$ of the ℓ -th Schur transform of $(s_j)_{j=0}^\infty$. A representation of the resolvent matrix of the matricial truncated Stieltjes moment problem in terms of the Schur transformed moment sequence is obtained in Section 7. In Section 8, interrelations between the polynomials $P_{k,j}^{(1)}$ and $Q_{k,j}^{(1)}$ constructed from the Schur transformed moment sequence with the polynomials $P_{k,j}$ and $Q_{k,j}$ corresponding to the original moment sequence are presented. A non-negative Hermitian measure, for which the matrix polynomials of the second kind are orthogonal matrix polynomials, is attained.

2. Stieltjes parametrization

With later applications to the matrix version of the Stieltjes moment problem in mind, a particular inner parametrization, called Stieltjes parametrization, for matrix sequences was developed in [25]. First we are going to recall the definition of the Stieltjes parametrization of a sequence of complex $p \times q$ matrices. To prepare this notion, we need some further matrices built from the given data. If $A \in \mathbb{C}^{p \times q}$, then a unique matrix $G \in \mathbb{C}^{q \times p}$ exists which satisfies the four equations $AGA = A$, $GAG = G$, $(AG)^* = AG$ and $(GA)^* = GA$. This matrix G is called the *Moore-Penrose inverse* of A and is denoted by A^\dagger .

Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then, let

$$y_{\ell,m} := \begin{bmatrix} s_\ell \\ s_{\ell+1} \\ \vdots \\ s_m \end{bmatrix} \quad \text{and} \quad z_{\ell,m} := [s_\ell, s_{\ell+1}, \dots, s_m] \quad (2.1)$$

for all $\ell, m \in \mathbb{N}_0$ with $\ell \leq m \leq \kappa$. We use the notation

$$L_{1,0} := s_0 \quad \text{and} \quad L_{1,n} := s_{2n} - z_{n,2n-1} H_{1,n-1}^\dagger y_{n,2n-1} \quad (2.2)$$

for all $n \in \mathbb{N}$ with $2n \leq \kappa$, and the notation

$$L_{2,0} := s_1 \quad \text{and} \quad L_{2,n} := s_{2n+1} - z_{n+1,2n} H_{2,n-1}^\dagger y_{n+1,2n} \quad (2.3)$$

for all $n \in \mathbb{N}$ with $2n+1 \leq \kappa$. Observe that for $n \geq 1$ the matrix $L_{1,n}$ is the Schur complement $H_{1,n}/H_{1,n-1}$ of $H_{1,n-1}$ in the block Hankel matrix $H_{1,n} = \begin{bmatrix} H_{1,n-1} & y_{n,2n-1} \\ z_{n,2n-1} & s_{2n} \end{bmatrix}$ corresponding to the sequence $(s_j)_{j=0}^\kappa$, whereas the matrix $L_{2,n}$ is the Schur complement $H_{2,n}/H_{2,n-1}$ of $H_{2,n-1}$ in the block Hankel matrix $H_{2,n} = \begin{bmatrix} H_{2,n-1} & y_{n+1,2n} \\ z_{n+1,2n} & s_{2n+1} \end{bmatrix}$ corresponding to the sequence $(s_{2,j})_{j=0}^{\kappa-1}$ defined by

$$s_{2,j} := s_{j+1}. \quad (2.4)$$

2. Stieltjes parametrization

The sequence $(s_{2,j})_{j=0}^{\kappa-1}$ coincides with the right-sided version of the sequence in [25, Definition 2.1, p. 217] for $\alpha = 0$.

Now we are able to recall the notion of Stieltjes parametrization which was introduced in [25, Definition 4.2, p. 223] as *right-sided α -Stieltjes parametrization* denoted by $(Q_j)_{j=0}^\kappa$ in a more general context related to a semi-infinite interval $[\alpha, \infty)$. There one can find further details.

Definition 2.1. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then the sequence $(\mathfrak{s}_j)_{j=0}^\kappa$ given by $\mathfrak{s}_{2k} := L_{1,k}$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$, and by $\mathfrak{s}_{2k+1} := L_{2,k}$ for all $k \in \mathbb{N}_0$ with $2k+1 \leq \kappa$ is called the *Stieltjes parametrization* of $(s_j)_{j=0}^\kappa$.

There is a one-to-one correspondence between a sequence $(s_j)_{j=0}^\kappa$ of complex $p \times q$ matrices and its Stieltjes parametrization $(\mathfrak{s}_j)_{j=0}^\kappa$ (see [25, Remark 4.3, p. 224]). In particular, the original sequence can be explicitly reconstructed from its Stieltjes parametrization. Let $\mathcal{N}(A)$ be the null space of a complex matrix A .

Proposition 2.2 (see [25, Theorem 4.12(b), p. 225]). *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices with Stieltjes parametrization $(\mathfrak{s}_j)_{j=0}^\kappa$. Then $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^{\geq}$ if and only if $\mathfrak{s}_j \in \mathbb{C}^{q \times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and $\mathcal{N}(\mathfrak{s}_j) \subseteq \mathcal{N}(\mathfrak{s}_{j+1})$ for all $j \in \mathbb{Z}_{0,\kappa-2}$.*

Proposition 2.3 (see [25, Theorem 4.12(c), p. 225]). *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices with Stieltjes parametrization $(\mathfrak{s}_j)_{j=0}^\kappa$. Then $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^{\geq,e}$ if and only if $\mathfrak{s}_j \in \mathbb{C}^{q \times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and $\mathcal{N}(\mathfrak{s}_j) \subseteq \mathcal{N}(\mathfrak{s}_{j+1})$ for all $j \in \mathbb{Z}_{0,\kappa-1}$.*

Now we introduce an important subclass of the class of Stieltjes non-negative definite sequences. More precisely, we turn our attention to some subclass of $\mathcal{K}_{q,\kappa}^{\geq}$, which is characterized by stronger positivity properties. For all $n \in \mathbb{N}_0$, let $\mathcal{K}_{q,2n+1}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices, such that $H_{1,n}$ and $H_{2,n}$ are both positive Hermitian. For all $n \in \mathbb{N}_0$, let $\mathcal{K}_{q,2n}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices, such that $H_{1,n}$ is positive Hermitian and, in the case $n \geq 1$, furthermore $H_{2,n-1}$ is positive Hermitian. Let $\mathcal{K}_{q,\infty}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^\infty$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^m$ belongs to $\mathcal{K}_{q,m}^{\geq}$ for all $m \in \mathbb{N}_0$. For all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we call a sequence $(s_j)_{j=0}^\kappa$ *Stieltjes positive definite* if it belongs to $\mathcal{K}_{q,\kappa}^{\geq}$. For $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, we have $\mathcal{K}_{q,\kappa}^{\geq} \subseteq \mathcal{K}_{q,\kappa}^{\geq,e} \subseteq \mathcal{K}_{q,\kappa}^{\geq}$ (see [28, Proposition 3.8, p. 12]). In view of Theorems 1.1 and 1.2 we obtain then:

Remark 2.4. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^{\geq}$. Then, $\mathcal{M}_{\geq}^q[[0, \infty); (s_j)_{j=0}^\kappa, =] \neq \emptyset$ and, in the case $\kappa < \infty$, furthermore $\mathcal{M}_{\geq}^q[[0, \infty); (s_j)_{j=0}^\kappa, \leq] \neq \emptyset$.

If $m \in \mathbb{N}_0$ and $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^{\geq}$, the associated truncated moment problems $\mathbf{M}[[0, \infty); (s_j)_{j=0}^m, =]$ and $\mathbf{M}[[0, \infty); (s_j)_{j=0}^m, \leq]$ have infinitely many solutions. Since every principal submatrix of a positive Hermitian matrix is again positive Hermitian, we can easily see:

Remark 2.5. Let $\kappa \in \mathbb{N} \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^{\geq}$, then $(s_{2,j})_{j=0}^{\kappa-1} \in \mathcal{K}_{q,\kappa-1}^{\geq}$.

2. Stieltjes parametrization

Proposition 2.6 (see [25, Theorem 4.12(d), p. 225]). *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices with Stieltjes parametrization $(\mathfrak{k}_j)_{j=0}^\kappa$. Then $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^>$ if and only if $\mathfrak{k}_j \in \mathbb{C}_{>}^{q \times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$.*

Based on the matrices defined via (2.2) and (2.3), we now introduce a further important subclass of $\mathcal{K}_{q,\kappa}^>$. Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^>$. Then $(s_j)_{j=0}^m$ is called *completely degenerate* if $L_{1,n} = 0_{q \times q}$ in the case $m = 2n$ with some $n \in \mathbb{N}_0$ or if $L_{2,n} = 0_{q \times q}$ in the case $m = 2n + 1$ with some $n \in \mathbb{N}_0$. The set $\mathcal{K}_{q,m}^{\geq, \text{cd}}$ of all completely degenerate sequences belonging to $\mathcal{K}_{q,m}^{\geq, \text{e}}$ (see [25, Proposition 5.9, p. 231]). The moment problem $\mathbf{M}[[0, \infty); (s_j)_{j=0}^m, =]$ has a unique solution if and only if $(s_j)_{j=0}^m$ belongs to $\mathcal{K}_{q,m}^{\geq, \text{cd}}$ (see [27, Theorem 13.3, p. 53]).

Proposition 2.7 (cf. [25, Proposition 5.3, p. 229]). *Let $m \in \mathbb{N}_0$ and $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^>$ with Stieltjes parametrization $(\mathfrak{k}_j)_{j=0}^m$. Then $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^{\geq, \text{cd}}$ if and only if $\mathfrak{k}_m = 0_{q \times q}$.*

If $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^{\geq, \text{e}}$ with Stieltjes parametrization $(\mathfrak{k}_j)_{j=0}^m$, then from [20, Lemmata 4.15 and 4.16] one can easily see that $(s_j)_{j=0}^m$ belongs to $\mathcal{K}_{q,m}^{\geq, \text{cd}}$ if and only if there is some $\ell \in \mathbb{Z}_{0,m}$ such that $\mathfrak{k}_\ell = 0_{q \times q}$.

Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Then $(s_j)_{j=0}^\infty$ is said to be *completely degenerate* if there is some $m \in \mathbb{N}_0$ such that $(s_j)_{j=0}^m$ is a completely degenerate Stieltjes non-negative definite sequence. By $\mathcal{K}_{q,\infty}^{\geq, \text{cd}}$ we denote the set of all completely degenerate Stieltjes non-negative definite sequences $(s_j)_{j=0}^\infty$ of complex $q \times q$ matrices.

Proposition 2.8 (cf. [25, Corollary 5.4, p. 230]). *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ with Stieltjes parametrization $(\mathfrak{k}_j)_{j=0}^\infty$. Then $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^{\geq, \text{cd}}$ if and only if there exists some $m \in \mathbb{N}_0$ with $\mathfrak{k}_m = 0_{q \times q}$.*

The sequence $(s_j)_{j=0}^\infty$ is called *completely degenerate of order m* if $(s_j)_{j=0}^m$ is completely degenerate. By $\mathcal{K}_{q,\infty}^{\geq, \text{cd}, m}$ we denote the set of all Stieltjes non-negative definite sequences $(s_j)_{j=0}^\infty$ from $\mathbb{C}^{q \times q}$ which are completely degenerate of order m . If $m \in \mathbb{N}_0$ and $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^{\geq, \text{cd}, m}$, then $(s_j)_{j=0}^\ell \in \mathcal{K}_{q,\ell}^{\geq, \text{cd}}$ for each $\ell \in \mathbb{Z}_{m,\infty}$ (cf. [25, Lemma 5.5, p. 230]).

Definition 2.9. Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m$ be a sequence of complex $p \times q$ matrices. Let the sequence $(\mathfrak{l}_j)_{j=0}^\infty$ be given by

$$\mathfrak{l}_j := \begin{cases} \mathfrak{k}_j, & \text{if } j \leq m \\ 0_{p \times q}, & \text{if } j > m \end{cases}.$$

Then, we call the unique sequence $(s_j^\circ)_{j=0}^\infty$ with Stieltjes parametrization $(\mathfrak{l}_j)_{j=0}^\infty$ the *zero Stieltjes parameter extension of $(s_j)_{j=0}^m$* .

Lemma 2.10. *Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^{\geq, \text{e}}$. Denote by $(s_j^\circ)_{j=0}^\infty$ the zero Stieltjes parameter extension of $(s_j)_{j=0}^m$. Then, $s_j^\circ = s_j$ for all $j \in \mathbb{Z}_{0,m}$ and $(s_j^\circ)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^{\geq, \text{cd}, m+1}$.*

3. Dyukarev-Stieltjes parametrization

Proof. Denote by $(\mathfrak{k}_j)_{j=0}^m$ the Stieltjes parametrization of $(s_j)_{j=0}^m$ and by $(\mathfrak{l}_j)_{j=0}^\infty$ the Stieltjes parametrization of $(s_j^\circ)_{j=0}^\infty$. Since $\mathfrak{l}_j = \mathfrak{k}_j$ holds true for all $j \in \mathbb{Z}_{0,m}$, we have $s_j^\circ = s_j$ for all $j \in \mathbb{Z}_{0,m}$. Using Propositions 2.3 and 2.2 we can conclude furthermore $(s_j^\circ)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^{\geq}$. In view of $\mathfrak{l}_{m+1} = 0_{q \times q}$ and Proposition 2.7, thus $(s_j^\circ)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^{\geq, \text{cd}, m+1}$ follows. \square

Lemma 2.11. *Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^{\geq, \text{cd}, m+1}$. Denote by $(s_j^\circ)_{j=0}^\infty$ the zero Stieltjes parameter extension of $(s_j)_{j=0}^m$. Then, $s_j^\circ = s_j$ for all $j \in \mathbb{N}_0$.*

Proof. Denote by $(\mathfrak{k}_j)_{j=0}^\infty$ the Stieltjes parametrization of $(s_j)_{j=0}^\infty$ and by $(\mathfrak{l}_j)_{j=0}^\infty$ the Stieltjes parametrization of $(s_j^\circ)_{j=0}^\infty$. Since $(\mathfrak{k}_j)_{j=0}^m$ is then the Stieltjes parametrization of $(s_j)_{j=0}^m$, we have by definition $\mathfrak{l}_j = \mathfrak{k}_j$ for all $j \in \mathbb{Z}_{0,m}$. Because of $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^{\geq, \text{cd}, m+1}$, the sequence $(s_j)_{j=0}^{m+1}$ belongs to $\mathcal{K}_{q, m+1}^{\geq, \text{cd}}$. Thus, Proposition 2.7 yields $\mathfrak{k}_{m+1} = 0_{q \times q}$. From Proposition 2.2, we obtain then $\mathfrak{k}_j = 0_{q \times q}$ for all $j \in \mathbb{Z}_{m+2, \infty}$. By definition, we have furthermore $\mathfrak{l}_j = 0_{q \times q}$ for all $j \in \mathbb{Z}_{m+1, \infty}$. Hence, $\mathfrak{l}_j = \mathfrak{k}_j$ for all $j \in \mathbb{Z}_{m+1, \infty}$. We have shown that the Stieltjes parametrizations of $(s_j)_{j=0}^\infty$ and $(s_j^\circ)_{j=0}^\infty$ coincide, which completes the proof. \square

3. Dyukarev-Stieltjes parametrization

In [19] Yu. M. Dyukarev studied the moment problem $\mathbf{M}[[0, \infty); (s_j)_{j=0}^\infty, =]$. One of his main results (see [19, Theorem 8, p. 78]) is a generalization of a classical criterion due to Stieltjes [37, 38] for the indeterminacy of this moment problem. In order to find an appropriate matricial version of Stieltjes' indeterminacy criterion Yu. M. Dyukarev had to look for a convenient matricial generalization of the parameter sequences which Stieltjes obtained from the consideration of particular continued fractions associated with the sequence $(s_j)_{j=0}^\infty$. In this way, Yu. M. Dyukarev found an interesting inner parametrization of sequences belonging to $\mathcal{K}_{q,\infty}^>$. The main theme of this section is to recall some interrelations obtained in [26, §8] between Yu. M. Dyukarev's parametrization and the Stieltjes parametrization introduced in Definition 2.1.

The notations I_q and $0_{p \times q}$ stand for the identity matrix in $\mathbb{C}^{q \times q}$ and for the zero matrix in $\mathbb{C}^{p \times q}$, resp. If $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^>$, then the matrix $H_{1,k}$ is positive Hermitian and, in particular, invertible for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$, and the matrix $H_{2,k}$ is positive Hermitian and, in particular, invertible for all $k \in \mathbb{N}_0$ with $2k+1 \leq \kappa$. Let

$$v_{q,0} := I_q \quad \text{and} \quad v_{q,k} := \begin{bmatrix} I_q \\ 0_{kq \times q} \end{bmatrix} \quad (3.1)$$

for all $k \in \mathbb{N}$. The following construction of a pair of sequences of $q \times q$ matrices associated with a Stieltjes positive definite sequence goes back to Yu. M. Dyukarev [19, p. 77]:

Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^>$. Then let

$$\mathbf{M}_0 := s_0^{-1} \quad (3.2)$$

3. Dyukarev-Stieltjes parametrization

and, in the case $\kappa \geq 1$, let

$$\mathbf{L}_0 := s_0 s_1^{-1} s_0. \quad (3.3)$$

Furthermore, let

$$\mathbf{M}_k := v_{q,k}^* H_{1,k}^{-1} v_{q,k} - v_{q,k-1}^* H_{1,k-1}^{-1} v_{q,k-1} \quad (3.4)$$

for all $k \in \mathbb{N}$ with $2k \leq \kappa$, and let

$$\mathbf{L}_k := y_{0,k}^* H_{2,k}^{-1} y_{0,k} - y_{0,k-1}^* H_{2,k-1}^{-1} y_{0,k-1} \quad (3.5)$$

for all $k \in \mathbb{N}$ with $2k + 1 \leq \kappa$.

Obviously, for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$, the matrix \mathbf{M}_k only depends on the matrices s_0, \dots, s_{2k} , and, for all $k \in \mathbb{N}_0$ with $2k + 1 \leq \kappa$, the matrix \mathbf{L}_k only depends on the matrices $s_0, s_1, \dots, s_{2k+1}$.

Definition 3.1. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$, then the ordered pair $[(\mathbf{L}_k)_{k=0}^\infty, (\mathbf{M}_k)_{k=0}^\infty]$ is called the *Dyukarev-Stieltjes parametrization* (shortly *DS-parametrization*) of $(s_j)_{j=0}^\infty$.

It should be mentioned that, for a given sequence $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$, Yu. M. Dyukarev [19] treated the moment problem $\mathbf{M}[[0, \infty); (s_j)_{j=0}^\infty, =]$ by approximation through the sequence $(\mathbf{M}[[0, \infty); (s_j)_{j=0}^k, \le])_{k \in \mathbb{N}_0}$ of truncated moment problems. One of his central results [19, Theorem 7, p. 77] shows that the resolvent matrices for the truncated moment problems can be multiplicatively decomposed into elementary factors which are determined by the corresponding first sections of the DS-parametrization of $(s_j)_{j=0}^\infty$.

If $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ with DS-parametrization $[(\mathbf{L}_k)_{k=0}^\infty, (\mathbf{M}_k)_{k=0}^\infty]$, then, in view of [19, Theorem 7, p. 77], the matrices \mathbf{L}_k and \mathbf{M}_k are positive Hermitian for all $k \in \mathbb{N}_0$. According to [26, Proposition 8.26, p. 3923], every sequence $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ can be recursively reconstructed from its DS-parametrization $[(\mathbf{L}_k)_{k=0}^\infty, (\mathbf{M}_k)_{k=0}^\infty]$. A similar result for the truncated DS-parametrization $[(\mathbf{L}_k)_{k=0}^{m-1}, (\mathbf{M}_k)_{k=0}^m]$ (resp. $[(\mathbf{L}_k)_{k=0}^m, (\mathbf{M}_k)_{k=0}^m]$) was obtained in [13, Proposition 4.9, p. 68]. Furthermore, [26, Proposition 8.27, p. 3924] shows that each pair $[(\mathbf{L}_k)_{k=0}^\infty, (\mathbf{M}_k)_{k=0}^\infty]$ of sequences of positive Hermitian complex $q \times q$ matrices is the DS-parametrization of some sequence $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Hence, the DS-parametrization establishes a one-to-one correspondence between Stieltjes positive definite sequences $(s_j)_{j=0}^\infty$ and ordered pairs $[(\mathbf{L}_k)_{k=0}^\infty, (\mathbf{M}_k)_{k=0}^\infty]$ of sequences of positive Hermitian complex $q \times q$ matrices.

In [26, Proposition 8.30, p. 3925] it was shown that in the scalar case the DS-parametrization of a Stieltjes positive definite sequence coincides with the classical parameters used by Stieltjes [37, 38] to formulate his indeterminacy criterion. We mention that M. G. Kreĭn was able to find a mechanical interpretation for Stieltjes' investigations on continued fractions (see Gantmacher/Kreĭn [31, Anhang 2] or Akhiezer [3, Appendix]). Against to the background of his mechanical interpretation M. G. Kreĭn divided Stieltjes' original parameters into two groups which play the roles of lengths and masses, respectively. Now we want to recall the concrete definition of these parameters (see Kreĭn/Nudel'man [35, Chapter V, formula (6.1)]) and their connection to the DS-parametrization.

3. Dyukarev-Stieltjes parametrization

Definition 3.2. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{1,\infty}^>$ and let $\Delta_n := \det H_{1,n}$ and $\Delta_n^{(1)} := \det H_{2,n}$ for all $n \in \mathbb{N}_0$. Let

$$l_k := \frac{\Delta_k^2}{\Delta_k^{(1)} \Delta_{k-1}^{(1)}} \quad \text{and} \quad m_k := \frac{(\Delta_{k-1}^{(1)})^2}{\Delta_k \Delta_{k-1}}$$

for all $k \in \mathbb{N}_0$, where $\Delta_{-1} := 1$ and $\Delta_{-1}^{(1)} := 1$. Then the ordered pair $[(l_k)_{k=0}^\infty, (m_k)_{k=0}^\infty]$ is called the *Krein-Stieltjes parametrization* of $(s_j)_{j=0}^\infty$.

Proposition 3.3 (see [26, Proposition 8.30, p. 3925]). *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{1,\infty}^>$ with Krein-Stieltjes parametrization $[(l_k)_{k=0}^\infty, (m_k)_{k=0}^\infty]$ and DS-parametrization $[(\mathbf{L}_k)_{k=0}^\infty, (\mathbf{M}_k)_{k=0}^\infty]$. Then $l_k = \mathbf{L}_k$ and $m_k = \mathbf{M}_k$ for all $k \in \mathbb{N}_0$.*

Now we recall the connection between the DS-parametrization of a Stieltjes positive definite sequence and its Stieltjes parametrization. If $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ with Stieltjes parametrization $(\mathfrak{k}_j)_{j=0}^\infty$, then, in view of Proposition 2.6, the matrices \mathfrak{k}_j are positive Hermitian and, in particular, invertible for all $j \in \mathbb{N}_0$.

Theorem 3.4 (see [26, Theorem 8.22, p. 3921]). *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ with Stieltjes parametrization $(\mathfrak{k}_j)_{j=0}^\infty$ and DS-parametrization $[(\mathbf{L}_k)_{k=0}^\infty, (\mathbf{M}_k)_{k=0}^\infty]$. Then*

$$\mathbf{L}_k = \left(\prod_{j=0}^{\overrightarrow{k}} \mathfrak{k}_{2j} \mathfrak{k}_{2j+1}^{-1} \right) \mathfrak{k}_{2k+1} \left(\prod_{j=0}^{\overrightarrow{k}} \mathfrak{k}_{2j} \mathfrak{k}_{2j+1}^{-1} \right)^*$$

and

$$\mathbf{M}_k = \begin{cases} \mathfrak{k}_0^{-1}, & \text{if } k = 0 \\ \left(\prod_{j=0}^{\overrightarrow{k-1}} \mathfrak{k}_{2j}^{-1} \mathfrak{k}_{2j+1} \right) \mathfrak{k}_{2k}^{-1} \left(\prod_{j=0}^{\overrightarrow{k-1}} \mathfrak{k}_{2j}^{-1} \mathfrak{k}_{2j+1} \right)^*, & \text{if } k \geq 1 \end{cases}$$

for all $k \in \mathbb{N}_0$.

Theorem 3.5 (see [26, Theorem 8.24, p. 3923] and [13, Corollary 4.10, p. 72]). *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ with Stieltjes parametrization $(\mathfrak{k}_j)_{j=0}^\infty$ and DS-parametrization $[(\mathbf{L}_k)_{k=0}^\infty, (\mathbf{M}_k)_{k=0}^\infty]$. Then*

$$\mathfrak{k}_{2k} = \begin{cases} \mathbf{M}_0^{-1}, & \text{if } k = 0 \\ \left(\prod_{j=0}^{\overrightarrow{k-1}} \mathbf{M}_j \mathbf{L}_j \right)^{-*} \mathbf{M}_k^{-1} \left(\prod_{j=0}^{\overrightarrow{k-1}} \mathbf{M}_j \mathbf{L}_j \right)^{-1}, & \text{if } k \geq 1 \end{cases}$$

and

$$\mathfrak{k}_{2k+1} = \left(\prod_{j=0}^{\overrightarrow{k}} \mathbf{M}_j \mathbf{L}_j \right)^{-*} \mathbf{L}_k \left(\prod_{j=0}^{\overrightarrow{k}} \mathbf{M}_j \mathbf{L}_j \right)^{-1}$$

for all $k \in \mathbb{N}_0$.

4. Parametrization of all solutions in the non-degenerate case

In this section we recall a parametrization of the solution set of the moment problem $M[[0, \infty); (s_j)_{j=0}^m, \leq]$ for the so-called non-degenerate case. Let \mathcal{S}_q be the set of all holomorphic functions $F: \mathbb{C} \setminus [0, \infty) \rightarrow \mathbb{C}^{q \times q}$ which satisfy the following conditions:

- (I) The matrix $\operatorname{Im} F(w)$ is non-negative Hermitian for all $w \in \mathbb{C}$ with $\operatorname{Im} w > 0$.
- (II) The matrix $F(x)$ is non-negative Hermitian for all $x \in (-\infty, 0)$.

Further, let $\mathcal{S}_{0,q}$ be the set of all $S \in \mathcal{S}_q$ such that

$$\sup_{y \in [1, \infty)} y \|S(iy)\|_{\mathbb{E}} < \infty,$$

where $\|\cdot\|_{\mathbb{E}}$ is the Euclidean matrix norm. We have the following integral representation for functions belonging to $\mathcal{S}_{0,q}$:

Theorem 4.1 (cf. [27, Theorem 5.1, p. 19]). *Let $S: \mathbb{C} \setminus [0, \infty) \rightarrow \mathbb{C}^{q \times q}$.*

- (a) *If $S \in \mathcal{S}_{0,q}$, then there exists a unique non-negative Hermitian measure $\sigma \in \mathcal{M}_{\geq}^q([0, \infty))$ such that*

$$S(z) = \int_{[0, \infty)} \frac{1}{t - z} \sigma(dt) \quad (4.1)$$

for all $z \in \mathbb{C} \setminus [0, \infty)$.

- (b) *If there exists a non-negative Hermitian measure $\sigma \in \mathcal{M}_{\geq}^q([0, \infty))$ such that S can be represented via (4.1) for all $z \in \mathbb{C} \setminus [0, \infty)$, then S belongs to $\mathcal{S}_{0,q}$.*

If σ is a measure belonging to $\mathcal{M}_{\geq}^q([0, \infty))$, then we will call the matrix-valued function $S: \mathbb{C} \setminus [0, \infty) \rightarrow \mathbb{C}^{q \times q}$ which is given for all $z \in \mathbb{C} \setminus [0, \infty)$ by (4.1) the *Stieltjes transform* of σ and write S_σ for S . If $S \in \mathcal{S}_{0,q}$, then the unique measure σ which belongs to $\mathcal{M}_{\geq}^q([0, \infty))$ and which fulfills (4.1) for all $z \in \mathbb{C} \setminus [0, \infty)$ is said to be the *Stieltjes measure* of S .

In view of Theorem 4.1, the moment problem $M[[0, \infty); (s_j)_{j=0}^m, \leq]$ can be reformulated:

Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{S}_{0,q}[(s_j)_{j=0}^m, \leq]$ of all $S \in \mathcal{S}_{0,q}$ with Stieltjes measure belonging to $\mathcal{M}_{\geq}^q([0, \infty); (s_j)_{j=0}^m, \leq]$.

Let the $2q \times 2q$ signature matrices J_q and \tilde{J}_q associated with the real and imaginary part be defined by

$$J_q := \begin{bmatrix} 0_{q \times q} & I_q \\ I_q & 0_{q \times q} \end{bmatrix} \quad \text{and} \quad \tilde{J}_q := \begin{bmatrix} 0_{q \times q} & iI_q \\ -iI_q & 0_{q \times q} \end{bmatrix}.$$

Let $\hat{\mathcal{S}}_q$ be the set of all ordered pairs (ϕ, ψ) of $\mathbb{C}^{q \times q}$ -valued functions ϕ and ψ which are meromorphic in $\mathbb{C} \setminus [0, \infty)$ and for which there exists a discrete subset $\mathcal{D}_{\phi, \psi}$ of $\mathbb{C} \setminus [0, \infty)$ such that the following conditions are fulfilled:

4. Parametrization of all solutions in the non-degenerate case

(III) The functions ϕ and ψ are both holomorphic in $\mathbb{C} \setminus ([0, \infty) \cup \mathcal{D}_{\phi, \psi})$.

(IV) $\text{rank} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = q$ for all $z \in \mathbb{C} \setminus ([0, \infty) \cup \mathcal{D}_{\phi, \psi})$.

(V) For all $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D}_{\phi, \psi})$,

$$\frac{1}{\text{Im } z} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \in \mathbb{C}_{\geq}^{q \times q}.$$

(VI) For all $z \in \mathbb{C} \setminus \mathcal{D}_{\phi, \psi}$ with $\text{Re } z < 0$,

$$\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^* J_q \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \in \mathbb{C}_{\geq}^{q \times q}.$$

Two pairs $(\phi_1, \psi_1), (\phi_2, \psi_2) \in \hat{\mathcal{S}}_q$ are called *equivalent*, if there exists a $\mathbb{C}^{q \times q}$ -valued function η which is meromorphic in $\mathbb{C} \setminus [0, \infty)$ such that $\det \eta$ does not vanishing identically and

$$\begin{bmatrix} \phi_2 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \psi_1 \end{bmatrix} \eta.$$

For all $n \in \mathbb{N}_0$, let

$$T_{q,n} := [\delta_{j,k+1} I_q]_{j,k=0}^n$$

and let $R_{q,n}: \mathbb{C} \rightarrow \mathbb{C}^{(n+1)q \times (n+1)q}$ be defined by

$$R_{q,n}(z) := (I_{(n+1)q} - zT_{q,n})^{-1}.$$

Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Then let

$$u_0 := 0_{q \times q} \quad \text{and} \quad u_k := \begin{bmatrix} 0_{q \times q} \\ -y_{0,k-1} \end{bmatrix}$$

for all $k \in \mathbb{Z}_{1,\kappa+1}$. Now we suppose that $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{K}_{q,\kappa}^>$. Let $\alpha_n: \mathbb{C} \rightarrow \mathbb{C}^{q \times q}$ and $\gamma_n: \mathbb{C} \rightarrow \mathbb{C}^{q \times q}$ be defined by

$$\alpha_n(z) := I_q - zu_n^* [R_{q,n}(\bar{z})]^* H_{1,n}^{-1} v_{q,n} \quad (4.2)$$

and

$$\gamma_n(z) := -zv_{q,n}^* [R_{q,n}(\bar{z})]^* H_{1,n}^{-1} v_{q,n} \quad (4.3)$$

for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, and let $\beta_n: \mathbb{C} \rightarrow \mathbb{C}^{q \times q}$ and $\delta_n: \mathbb{C} \rightarrow \mathbb{C}^{q \times q}$ be defined by

$$\beta_n(z) := \begin{cases} 0_{q \times q}, & \text{if } n = 0 \\ y_{0,n-1}^* [R_{q,n-1}(\bar{z})]^* H_{2,n-1}^{-1} y_{0,n-1}, & \text{if } n \geq 1 \end{cases} \quad (4.4)$$

and

$$\delta_n(z) := \begin{cases} I_q, & \text{if } n = 0 \\ I_q + zv_{q,n-1}^* [R_{q,n-1}(\bar{z})]^* H_{2,n-1}^{-1} y_{0,n-1}, & \text{if } n \geq 1 \end{cases} \quad (4.5)$$

4. Parametrization of all solutions in the non-degenerate case

for all $n \in \mathbb{N}_0$ with $2n - 1 \leq \kappa$. Then

$$\begin{aligned} \alpha_0(z) &= I_q, & \beta_0(z) &= 0_{q \times q} \\ \gamma_0(z) &= -zs_0^{-1}, & \text{and} & \delta_0(z) = I_q \end{aligned}$$

for all $z \in \mathbb{C}$. Let

$$U_{2n} := \begin{bmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{bmatrix} \quad (4.6)$$

for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, and let

$$U_{2n+1} := \begin{bmatrix} \alpha_n & \beta_{n+1} \\ \gamma_n & \delta_{n+1} \end{bmatrix} \quad (4.7)$$

for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$.

Definition 4.2. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Then $(U_m)_{m=0}^\infty$ is called the *sequence of Dyukarev matrix polynomials associated with $(s_j)_{j=0}^\infty$* .

Now we are able to recall a parametrization of the set $\mathcal{S}_{0,q}[(s_j)_{j=0}^m, \leq]$ with pairs belonging to $\hat{\mathcal{S}}_q$:

Theorem 4.3 (see [12, Theorem 3.2, p. 9] or [16, 18]). *Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^>$. Let $U_m = \begin{bmatrix} A_m & B_m \\ C_m & D_m \end{bmatrix}$ be the $q \times q$ block representation of U_m . Then:*

- (a) *Let $(\phi, \psi) \in \hat{\mathcal{S}}_q$. Then $\det(C_m\phi + D_m\psi)$ does not vanish identically in $\mathbb{C} \setminus [0, \infty)$ and*

$$(A_m\phi + B_m\psi)(C_m\phi + D_m\psi)^{-1} \in \mathcal{S}_{0,q}[(s_j)_{j=0}^m, \leq].$$

- (b) *Let $(\phi_1, \psi_1), (\phi_2, \psi_2) \in \hat{\mathcal{S}}_q$ be such that*

$$(A_m\phi_1 + B_m\psi_1)(C_m\phi_1 + D_m\psi_1)^{-1} = (A_m\phi_2 + B_m\psi_2)(C_m\phi_2 + D_m\psi_2)^{-1}.$$

Then, (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are equivalent.

- (c) *Let $S \in \mathcal{S}_{0,q}[(s_j)_{j=0}^m, \leq]$. Then, there exists a pair $(\phi, \psi) \in \hat{\mathcal{S}}_q$ such that*

$$S = (A_m\phi + B_m\psi)(C_m\phi + D_m\psi)^{-1}.$$

If $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^>$, then let $\mathbb{M}_\kappa: \mathbb{C} \rightarrow \mathbb{C}^{2q \times 2q}$ be defined by

$$\mathbb{M}_\kappa(z) := \begin{bmatrix} I_q & 0_{q \times q} \\ -z\mathbf{M}_\kappa & I_q \end{bmatrix} \quad (4.8)$$

for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$, and let

$$\mathbb{L}_\kappa := \begin{bmatrix} I_q & \mathbf{L}_\kappa \\ 0_{q \times q} & I_q \end{bmatrix} \quad (4.9)$$

for all $k \in \mathbb{N}_0$ with $2k + 1 \leq \kappa$. We have the following factorization of U_m in a product of complex $2q \times 2q$ matrix polynomials of degree 1:

4. Parametrization of all solutions in the non-degenerate case

Proposition 4.4 (see [12, Formulas (4.11) and (4.12), p. 11] or [19, Theorem 7, p. 77]).
Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^>$, then

$$U_0 = \mathbb{M}_0 \quad \text{and} \quad U_{2n} = \left(\prod_{k=0}^{n-1} \mathbb{M}_k \mathbb{L}_k \right) \mathbb{M}_n$$

for all $n \in \mathbb{N}$ with $2n \leq \kappa$, and

$$U_{2n+1} = \prod_{k=0}^n (\mathbb{M}_k \mathbb{L}_k)$$

for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$.

Notation 4.5. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^>$. Let $P_{1,n}: \mathbb{C} \rightarrow \mathbb{C}^{q \times q}$ and $Q_{1,n}: \mathbb{C} \rightarrow \mathbb{C}^{q \times q}$ be defined by

$$P_{1,n}(z) := \begin{cases} I_q, & \text{if } n = 0 \\ v_{q,n}^*[R_{q,n}(\bar{z})]^* \begin{bmatrix} -H_{1,n-1}^{-1} y_{n,2n-1} \end{bmatrix}_{I_q}, & \text{if } n \geq 1 \end{cases}$$

and

$$Q_{1,n}(z) := \begin{cases} 0_{q \times q}, & \text{if } n = 0 \\ -u_n^*[R_{q,n}(\bar{z})]^* \begin{bmatrix} -H_{1,n-1}^{-1} y_{n,2n-1} \end{bmatrix}_{I_q}, & \text{if } n \geq 1 \end{cases}$$

for all $n \in \mathbb{N}_0$ with $2n-1 \leq \kappa$, and let $P_{2,n}: \mathbb{C} \rightarrow \mathbb{C}^{q \times q}$ and $Q_{2,n}: \mathbb{C} \rightarrow \mathbb{C}^{q \times q}$ be defined by

$$P_{2,n}(z) := \begin{cases} I_q, & \text{if } n = 0 \\ v_{q,n}^*[R_{q,n}(\bar{z})]^* \begin{bmatrix} -H_{2,n-1}^{-1} y_{n+1,2n} \end{bmatrix}_{I_q}, & \text{if } n \geq 1 \end{cases}$$

and

$$Q_{2,n}(z) := \begin{cases} s_0, & \text{if } n = 0 \\ z_{0,n}[R_{q,n}(\bar{z})]^* \begin{bmatrix} -H_{2,n-1}^{-1} y_{n+1,2n} \end{bmatrix}_{I_q}, & \text{if } n \geq 1 \end{cases}$$

for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$.

Definition 4.6. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Then $[(P_{1,k})_{k=0}^\infty, (Q_{1,k})_{k=0}^\infty, (P_{2,k})_{k=0}^\infty, (Q_{2,k})_{k=0}^\infty]$ is called the *Stieltjes quadruple associated with $(s_j)_{j=0}^\infty$* .

Remark 4.7. For all $n \in \mathbb{N}_0$ with $2n-1 \leq \kappa$, the functions $P_{1,n}$ and $Q_{1,n}$ are $q \times q$ matrix polynomials, where $P_{1,n}$ has degree n and leading coefficient I_q . Similarly, for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, the functions $P_{2,n}$ and $Q_{2,n}$ are $q \times q$ matrix polynomials, where $P_{2,n}$ has degree n and leading coefficient I_q . Furthermore, if $\sigma_1 \in \mathcal{M}_{\geq}^q[[0, \infty); (s_j)_{j=0}^\kappa, =]$, we have the orthogonality relations

$$\int_{[0, \infty)} [P_{1,m}(t)]^* \sigma_1(dt) [P_{1,n}(t)] = \begin{cases} 0_{q \times q}, & \text{if } m \neq n \\ L_{1,n}, & \text{if } m = n \end{cases}$$

5. The Schur transform

for all $m, n \in \mathbb{N}_0$ with $2n - 1 \leq \kappa$ and $2m - 1 \leq \kappa$, and

$$\int_{[0, \infty)} [P_{2,m}(t)]^* \sigma_2(dt) [P_{2,n}(t)] = \begin{cases} 0_{q \times q}, & \text{if } m \neq n \\ L_{2,n}, & \text{if } m = n \end{cases}$$

for all $m, n \in \mathbb{N}_0$ with $2n \leq \kappa$ and $2m \leq \kappa$, where $\sigma_2: \mathfrak{B}_{[0, \infty)} \rightarrow \mathbb{C}_{\geq}^{q \times q}$ is defined by $\sigma_2(B) := \int_{[0, \infty)} t \sigma_1(dt)$ and belongs to $\mathcal{M}_{\geq}^q[[0, \infty); (s_{2,j})_{j=0}^{\kappa-1}, =]$ (see (2.4) and (1.1)).

Remark 4.8 (see [12, Lemma 4.3, p. 11]). Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q, \kappa}^>$, then

$$P_{1,n}(0) = (-1)^n \left(\prod_{k=0}^{\overrightarrow{n-1}} \mathbf{M}_k \mathbf{L}_k \right)^{-1}$$

for all $n \in \mathbb{N}$ with $2n - 1 \leq \kappa$, and

$$Q_{2,n}(0) = (-1)^n \left[\left(\prod_{k=0}^{\overrightarrow{n-1}} \mathbf{M}_k \mathbf{L}_k \right) \mathbf{M}_n \right]^{-1}$$

for all $n \in \mathbb{N}$ with $2n \leq \kappa$.

We have the following representation of U_m in terms of the above introduced matrix polynomials:

Proposition 4.9 (see [12, Theorem 4.7, p. 68]). *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q, \kappa}^>$. For all $z \in \mathbb{C}$, then*

$$U_{2n}(z) = \begin{bmatrix} Q_{2,n}(z) & -Q_{1,n}(z) \\ -zP_{2,n}(z) & P_{1,n}(z) \end{bmatrix} \begin{bmatrix} [Q_{2,n}(0)]^{-1} & 0_{q \times q} \\ 0_{q \times q} & [P_{1,n}(0)]^{-1} \end{bmatrix}$$

for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, and

$$U_{2n+1}(z) = \begin{bmatrix} Q_{2,n}(z) & -Q_{1,n+1}(z) \\ -zP_{2,n}(z) & P_{1,n+1}(z) \end{bmatrix} \begin{bmatrix} [Q_{2,n}(0)]^{-1} & 0_{q \times q} \\ 0_{q \times q} & [P_{1,n+1}(0)]^{-1} \end{bmatrix}$$

for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$.

5. The Schur transform

In [28, §7] a transformation for sequences of complex $p \times q$ matrices was considered using the following concept of reciprocal sequences presented in [30]. The paper [30] deals with the question of invertibility as it applies to matrix sequences.

Definition 5.1. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. The sequence $(s_j^{\sharp})_{j=0}^{\kappa}$ given recursively by

$$s_0^{\sharp} := s_0^{\dagger} \quad \text{and} \quad s_j^{\sharp} := -s_0^{\dagger} \sum_{\ell=0}^{j-1} s_{j-\ell} s_{\ell}^{\sharp}$$

for all $j \in \mathbb{Z}_{1, \kappa}$ is called the *reciprocal sequence corresponding to $(s_j)_{j=0}^{\kappa}$* .

6. The DS-parametrization after Schur transformation

Definition 5.2. Let $\kappa \in \mathbb{N} \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then, the sequence $(s_j^{[1]})_{j=0}^{\kappa-1}$ given by

$$s_j^{[1]} := -s_0 s_{j+1}^\# s_0$$

is called the *first Schur transform* of $(s_j)_{j=0}^\kappa$.

Observe that this transformation coincides with the first α -Schur-transform from [28, Definition 7.1, p. 33] for $\alpha = 0$ and served together with its counterpart for matrix-valued functions in [28, 29] as elementary step of a Schur type algorithm to solve the truncated matricial moment problem on the semi-infinite interval $[\alpha, \infty)$.

Definition 5.3. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. The sequence $(s_j^{[0]})_{j=0}^\kappa$ given by $s_j^{[0]} := s_j$ for all $j \in \mathbb{Z}_{0,\kappa}$ is called the *0-th Schur transform* of $(s_j)_{j=0}^\kappa$. In the case $\kappa \geq 1$, for all $k \in \mathbb{Z}_{1,\kappa}$, the *k-th Schur transform* $(s_j^{[k]})_{j=0}^{\kappa-k}$ of $(s_j)_{j=0}^\kappa$ is recursively defined by

$$s_j^{[k]} := t_j^{[1]}$$

for all $j \in \mathbb{Z}_{0,\kappa-k}$, where $(t_j)_{j=0}^{\kappa-(k-1)}$ denotes the $(k-1)$ -th Schur transform of $(s_j)_{j=0}^\kappa$.

Proposition 5.4 (see [28, Theorem 8.10(c), p. 42]). *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^>$. Then $(s_j^{[k]})_{j=0}^{\kappa-k} \in \mathcal{K}_{q,\kappa-k}^>$ for all $k \in \mathbb{Z}_{0,\kappa}$.*

Proposition 5.5 (see [28, Theorem 8.10(d), p. 42]). *Let $m \in \mathbb{N}_0$ and $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^{\geq, \text{cd}, m}$. Then $(s_j^{[k]})_{j=0}^{\kappa-k} \in \mathcal{K}_{q,\infty}^{\geq, \text{cd}, \max\{0, m-k\}}$ for all $k \in \mathbb{Z}_{0,\kappa}$.*

Proposition 5.6 (see [28, Theorem 8.10(e), p. 42]). *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^{\geq, \text{cd}}$. Then $(s_j^{[k]})_{j=0}^{\kappa-k} \in \mathcal{K}_{q,\kappa-k}^{\geq, \text{cd}}$ for all $k \in \mathbb{Z}_{0,\kappa}$.*

Proposition 5.7. *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^>$ with Stieltjes parametrization $(\mathfrak{s}_j)_{j=0}^\kappa$, and let $k \in \mathbb{Z}_{0,\kappa}$. Then $(\mathfrak{s}_{k+j})_{j=0}^{\kappa-k}$ is the Stieltjes parametrization of $(s_j^{[k]})_{j=0}^{\kappa-k}$.*

Proof. According to [25, Proposition 2.20, p. 221] we have $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa}^{\geq, \text{e}}$. Hence, the application of [28, Theorem 9.26, p. 57] yields the assertion. \square

6. The DS-parametrization after Schur transformation

From now on we consider only infinite sequences $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Let $\ell \in \mathbb{N}_0$. According to Proposition 5.4, then the ℓ -th Schur transform $(s_j^{[\ell]})_{j=0}^\infty$ of $(s_j)_{j=0}^\infty$ belongs to $\mathcal{K}_{q,\infty}^>$. If X is an object build from the sequence $(s_j)_{j=0}^\infty$, then we will use the notation $X^{(\ell)}$ for this object build from the sequence $(s_j^{[\ell]})_{j=0}^\infty$, e. g., in view of (1.2), we have

$$H_{1,n}^{(\ell)} := [s_{j+k}^{[\ell]}]_{j,k=0}^n \quad \text{and} \quad H_{2,n}^{(\ell)} := [s_{j+k+1}^{[\ell]}]_{j,k=0}^n$$

for all $n \in \mathbb{N}_0$. From Definition 2.1 and Proposition 5.7 we see:

6. The DS-parametrization after Schur transformation

Remark 6.1. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Then $L_{1,k}^{(1)} = L_{2,k}$ and $L_{2,k}^{(1)} = L_{1,k+1}$ for all $k \in \mathbb{N}_0$.

In view of Proposition 5.4 and Definition 3.1, we are particularly able to introduce the following notation:

Notation 6.2. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ and let $\ell \in \mathbb{N}_0$. Then, we write $[(\mathbf{L}_k^{(\ell)})_{k=0}^\infty, (\mathbf{M}_k^{(\ell)})_{k=0}^\infty]$ for the DS-parametrization of the ℓ -th Schur transform of $(s_j)_{j=0}^\infty$.

From Theorem 3.4 and Proposition 5.7 we obtain then:

Lemma 6.3. *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ and let $\ell \in \mathbb{N}_0$, then*

$$\mathbf{L}_k^{(\ell)} = \left(\prod_{j=0}^{\overrightarrow{k}} \mathfrak{k}_{2j+\ell} \mathfrak{k}_{2j+\ell+1}^{-1} \right) \mathfrak{k}_{2k+\ell+1} \left(\prod_{j=0}^{\overrightarrow{k}} \mathfrak{k}_{2j+\ell} \mathfrak{k}_{2j+\ell+1}^{-1} \right)^*$$

and

$$\mathbf{M}_k^{(\ell)} = \begin{cases} \mathfrak{k}_\ell^{-1}, & \text{if } k = 0 \\ \left(\prod_{j=0}^{\overrightarrow{k-1}} \mathfrak{k}_{2j+\ell} \mathfrak{k}_{2j+\ell+1}^{-1} \right) \mathfrak{k}_{2k+\ell}^{-1} \left(\prod_{j=0}^{\overrightarrow{k-1}} \mathfrak{k}_{2j+\ell} \mathfrak{k}_{2j+\ell+1}^{-1} \right)^*, & \text{if } k \geq 1 \end{cases}$$

for all $k \in \mathbb{N}_0$.

Using Lemma 6.3 we conclude:

Lemma 6.4. *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ and let $\ell \in \mathbb{N}_0$, then*

$$\mathbf{L}_k^{(\ell)} = \left(\prod_{j=0}^{\overrightarrow{m-1}} \mathfrak{k}_{2j+\ell} \mathfrak{k}_{2j+\ell+1}^{-1} \right) \mathbf{L}_{k-m}^{(\ell+2m)} \left(\prod_{j=0}^{\overrightarrow{m-1}} \mathfrak{k}_{2j+\ell} \mathfrak{k}_{2j+\ell+1}^{-1} \right)^*$$

and

$$\mathbf{M}_k^{(\ell)} = \left(\prod_{j=0}^{\overrightarrow{m-1}} \mathfrak{k}_{2j+\ell} \mathfrak{k}_{2j+\ell+1}^{-1} \right) \mathbf{M}_{k-m}^{(\ell+2m)} \left(\prod_{j=0}^{\overrightarrow{m-1}} \mathfrak{k}_{2j+\ell} \mathfrak{k}_{2j+\ell+1}^{-1} \right)^*$$

for all $k \in \mathbb{N}$ and $m \in \mathbb{Z}_{1,k}$.

Lemma 6.5. *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ and let $\ell \in \mathbb{N}_0$, then*

$$\mathbf{L}_k^{(\ell+1)} = \mathfrak{k}_\ell \mathbf{M}_{k+1}^{(\ell)} \mathfrak{k}_\ell \quad \text{and} \quad \mathbf{M}_k^{(\ell+1)} = \mathfrak{k}_\ell^{-1} \mathbf{L}_k^{(\ell)} \mathfrak{k}_\ell^{-1}$$

for all $k \in \mathbb{N}_0$.

6. The DS-parametrization after Schur transformation

Proof. From Proposition 2.6 we know that the matrices \mathfrak{k}_j are Hermitian and invertible for all $j \in \mathbb{N}_0$. Using Lemma 6.3, we obtain thus

$$\begin{aligned}
\mathbf{L}_k^{(\ell+1)} &= \mathfrak{k}_\ell \mathfrak{k}_\ell^{-1} \mathbf{L}_k^{(\ell+1)} \mathfrak{k}_\ell^{-*} \mathfrak{k}_\ell^* \\
&= \mathfrak{k}_\ell \mathfrak{k}_\ell^{-1} \left(\prod_{j=0}^k \mathfrak{k}_{2j+\ell+1} \mathfrak{k}_{2j+\ell+2}^{-1} \right) \mathfrak{k}_{2k+\ell+2} \left(\prod_{j=0}^k \mathfrak{k}_{2j+\ell+1} \mathfrak{k}_{2j+\ell+2}^{-1} \right)^* \mathfrak{k}_\ell^{-*} \mathfrak{k}_\ell^* \\
&= \mathfrak{k}_\ell \left(\prod_{j=0}^k \mathfrak{k}_{2j+\ell}^{-1} \mathfrak{k}_{2j+\ell+1} \right) \mathfrak{k}_{2k+\ell+2}^{-1} \mathfrak{k}_{2k+\ell+2} \mathfrak{k}_{2k+\ell+2}^{-*} \left(\prod_{j=0}^k \mathfrak{k}_{2j+\ell}^{-1} \mathfrak{k}_{2j+\ell+1} \right)^* \mathfrak{k}_\ell^* \\
&= \mathfrak{k}_\ell \left(\prod_{j=0}^k \mathfrak{k}_{2j+\ell}^{-1} \mathfrak{k}_{2j+\ell+1} \right) \mathfrak{k}_{2(k+1)+\ell}^{-1} \left(\prod_{j=0}^k \mathfrak{k}_{2j+\ell}^{-1} \mathfrak{k}_{2j+\ell+1} \right)^* \mathfrak{k}_\ell = \mathfrak{k}_\ell \mathbf{M}_{k+1}^{(\ell)} \mathfrak{k}_\ell
\end{aligned}$$

for all $k \in \mathbb{N}_0$ and

$$\begin{aligned}
\mathbf{M}_0^{(\ell+1)} &= \mathfrak{k}_\ell^{-1} \mathfrak{k}_\ell \mathbf{M}_0^{(\ell+1)} \mathfrak{k}_\ell^* \mathfrak{k}_\ell^{-*} = \mathfrak{k}_\ell^{-1} \mathfrak{k}_\ell \mathfrak{k}_{\ell+1}^{-1} \mathfrak{k}_\ell^* \mathfrak{k}_\ell^{-*} = \mathfrak{k}_\ell^{-1} \mathfrak{k}_\ell \mathfrak{k}_{\ell+1}^{-1} \mathfrak{k}_{\ell+1} \mathfrak{k}_{\ell+1}^{-*} \mathfrak{k}_\ell^* \mathfrak{k}_\ell^{-*} \\
&= \mathfrak{k}_\ell^{-1} (\mathfrak{k}_\ell \mathfrak{k}_{\ell+1}^{-1}) \mathfrak{k}_{\ell+1} (\mathfrak{k}_\ell \mathfrak{k}_{\ell+1}^{-1})^* \mathfrak{k}_\ell^{-1} = \mathfrak{k}_\ell^{-1} \mathbf{L}_0^{(\ell)} \mathfrak{k}_\ell^{-1}
\end{aligned}$$

and furthermore

$$\begin{aligned}
\mathbf{M}_k^{(\ell+1)} &= \mathfrak{k}_\ell^{-1} \mathfrak{k}_\ell \mathbf{M}_k^{(\ell+1)} \mathfrak{k}_\ell^* \mathfrak{k}_\ell^{-*} \\
&= \mathfrak{k}_\ell^{-1} \mathfrak{k}_\ell \left(\prod_{j=0}^{k-1} \mathfrak{k}_{2j+\ell+1}^{-1} \mathfrak{k}_{2j+\ell+2} \right) \mathfrak{k}_{2k+\ell+1}^{-1} \left(\prod_{j=0}^{k-1} \mathfrak{k}_{2j+\ell+1}^{-1} \mathfrak{k}_{2j+\ell+2} \right)^* \mathfrak{k}_\ell^* \mathfrak{k}_\ell^{-*} \\
&= \mathfrak{k}_\ell^{-1} \mathfrak{k}_\ell \left(\prod_{j=0}^{k-1} \mathfrak{k}_{2j+\ell+1}^{-1} \mathfrak{k}_{2j+\ell+2} \right) \mathfrak{k}_{2k+\ell+1}^{-1} \mathfrak{k}_{2k+\ell+1} \mathfrak{k}_{2k+\ell+1}^{-*} \left(\prod_{j=0}^{k-1} \mathfrak{k}_{2j+\ell+1}^{-1} \mathfrak{k}_{2j+\ell+2} \right)^* \mathfrak{k}_\ell^* \mathfrak{k}_\ell^{-1} \\
&= \mathfrak{k}_\ell^{-1} \left(\prod_{j=0}^k \mathfrak{k}_{2j+\ell} \mathfrak{k}_{2j+\ell+1}^{-1} \right) \mathfrak{k}_{2k+\ell+1} \left(\prod_{j=0}^k \mathfrak{k}_{2j+\ell} \mathfrak{k}_{2j+\ell+1}^{-1} \right)^* \mathfrak{k}_\ell^{-1} = \mathfrak{k}_\ell^{-1} \mathbf{L}_k^{(\ell)} \mathfrak{k}_\ell^{-1}
\end{aligned}$$

for all $k \in \mathbb{N}$. □

In view of Definition 2.1 and (2.2), we obtain from Lemma 6.5 the following relation between the DS-parametrizations of a Stieltjes positive definite sequence and its first Schur transform:

Theorem 6.6. *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ with DS-parametrization $[(\mathbf{L}_k)_{k=0}^\infty, (\mathbf{M}_k)_{k=0}^\infty]$. Then, the DS-parametrization $[(\mathbf{L}_k^{(1)})_{k=0}^\infty, (\mathbf{M}_k^{(1)})_{k=0}^\infty]$ of $(s_j^{[1]})_{j=0}^\infty$ is given by*

$$\mathbf{L}_k^{(1)} = s_0 \mathbf{M}_{k+1} s_0 \quad \text{and} \quad \mathbf{M}_k^{(1)} = s_0^{-1} \mathbf{L}_k s_0^{-1}$$

for all $k \in \mathbb{N}_0$.

7. The resolvent matrix corresponding to a Schur transformed moment sequence

In view of Theorem 3.5 and [26, Remark 8.4, p. 24] we have:

Remark 7.1. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ with DS-parametrization $[(\mathbf{L}_k)_{k=0}^\infty, (\mathbf{M}_k)_{k=0}^\infty]$ and Stieltjes parametrization $(\mathfrak{t}_j)_{j=0}^\infty$, then

$$\prod_{j=0}^{\overrightarrow{m}} \mathfrak{t}_{2j}^{-1} \mathfrak{t}_{2j+1} = \left(\prod_{j=0}^{\overrightarrow{m}} \mathbf{M}_j \mathbf{L}_j \right)^{-1} \quad \text{and} \quad \prod_{j=0}^{\overrightarrow{m}} \mathfrak{t}_{2j} \mathfrak{t}_{2j+1}^{-1} = \left(\prod_{j=0}^{\overrightarrow{m}} \mathbf{M}_j \mathbf{L}_j \right)^*$$

for all $m \in \mathbb{N}_0$.

Lemma 7.2. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ with DS-parametrization $[(\mathbf{L}_k)_{k=0}^\infty, (\mathbf{M}_k)_{k=0}^\infty]$ and Stieltjes quadruple $[(P_{1,k})_{k=0}^\infty, (Q_{1,k})_{k=0}^\infty, (P_{2,k})_{k=0}^\infty, (Q_{2,k})_{k=0}^\infty]$. Then,

$$\mathbf{L}_k = [P_{1,m}(0)]^{-*} \mathbf{L}_{k-m}^{(2m)} [P_{1,m}(0)]^{-1} = Q_{2,m}(0) \mathbf{M}_{k-m}^{(2m+1)} [Q_{2,m}(0)]^*$$

and

$$\mathbf{M}_k = P_{1,m}(0) \mathbf{M}_{k-m}^{(2m)} [P_{1,m}(0)]^*, \quad \mathbf{M}_{k+1} = [Q_{2,m}(0)]^{-*} \mathbf{L}_{k-m}^{(2m+1)} [Q_{2,m}(0)]^{-1}$$

for all $k \in \mathbb{N}_0$ and $m \in \mathbb{Z}_{0,k}$.

Proof. Let $k \in \mathbb{N}_0$ and $m \in \mathbb{Z}_{0,k}$. In the case $m = 0$ all assertions follow from Notation 4.5 and Remark 6.6. Now suppose that $k \in \mathbb{N}$ and $m \in \mathbb{Z}_{1,k}$. From Remarks 7.1 and 4.8 we obtain

$$\prod_{j=0}^{\overrightarrow{m-1}} \mathfrak{t}_{2j}^{-1} \mathfrak{t}_{2j+1} = (-1)^m P_{1,m}(0) \quad \text{and} \quad \prod_{j=0}^{\overrightarrow{m-1}} \mathfrak{t}_{2j} \mathfrak{t}_{2j+1}^{-1} = (-1)^m [P_{1,m}(0)]^{-*}.$$

Hence, using Lemma 6.4 with $\ell = 0$, we get

$$\mathbf{L}_k = \left(\prod_{j=0}^{\overrightarrow{m-1}} \mathfrak{t}_{2j} \mathfrak{t}_{2j+1}^{-1} \right) \mathbf{L}_{k-m}^{(2m)} \left(\prod_{j=0}^{\overrightarrow{m-1}} \mathfrak{t}_{2j} \mathfrak{t}_{2j+1}^{-1} \right)^* = [P_{1,m}(0)]^{-*} \mathbf{L}_{k-m}^{(2m)} [P_{1,m}(0)]^{-1}$$

and

$$\mathbf{M}_k = \left(\prod_{j=0}^{\overrightarrow{m-1}} \mathfrak{t}_{2j}^{-1} \mathfrak{t}_{2j+1} \right) \mathbf{M}_{k-m}^{(2m)} \left(\prod_{j=0}^{\overrightarrow{m-1}} \mathfrak{t}_{2j} \mathfrak{t}_{2j+1}^{-1} \right)^* = P_{1,m}(0) \mathbf{M}_{k-m}^{(2m)} [P_{1,m}(0)]^*. \quad (7.1)$$

According to Lemma 6.5 we have

$$\mathbf{L}_{k-m}^{(2m+1)} = \mathfrak{t}_{2m} \mathbf{M}_{k-m+1}^{(2m)} \mathfrak{t}_{2m} \quad \text{and} \quad \mathbf{M}_{k-m}^{(2m+1)} = \mathfrak{t}_{2m}^{-1} \mathbf{L}_{k-m}^{(2m)} \mathfrak{t}_{2m}^{-1}.$$

7. The resolvent matrix corresponding to a Schur transformed moment sequence

Theorem 3.5 and Remark 4.8 yield

$$\mathfrak{L}_{2m} = \left(\prod_{j=0}^{m-1} \mathbf{M}_j \mathbf{L}_j \right)^{-*} \mathbf{M}_m^{-1} \left(\prod_{j=0}^{m-1} \mathbf{M}_j \mathbf{L}_j \right)^{-1} = [P_{1,m}(0)]^* Q_{2,m}(0).$$

Thus, we get

$$\begin{aligned} [P_{1,m}(0)]^{-*} \mathbf{L}_{k-m}^{(2m)} [P_{1,m}(0)]^{-1} &= [P_{1,m}(0)]^{-*} \mathfrak{L}_{2m} \mathbf{M}_{k-m}^{(2m+1)} \mathfrak{L}_{2m}^* [P_{1,m}(0)]^{-1} \\ &= Q_{2,m}(0) \mathbf{M}_{k-m}^{(2m+1)} [Q_{2,m}(0)]^* \end{aligned}$$

and, using the already proved identity (7.1) with $k+1$ instead of k , furthermore

$$\begin{aligned} \mathbf{M}_{k+1} &= P_{1,m}(0) \mathbf{M}_{k-m+1}^{(2m)} [P_{1,m}(0)]^* = P_{1,m}(0) \mathfrak{L}_{2m}^{-*} \mathbf{L}_{k-m}^{(2m+1)} \mathfrak{L}_{2m}^{-1} [P_{1,m}(0)]^* \\ &= [Q_{2,m}(0)]^{-*} \mathbf{L}_{k-m}^{(2m+1)} [Q_{2,m}(0)]^{-1}. \quad \square \end{aligned}$$

Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Then, for all $n \in \mathbb{N}_0$, the matrices $P_{1,n}(0)$ and $Q_{2,n}(0)$ are invertible according to Remark 4.8 and Notation 4.5. Thus, for all $n \in \mathbb{N}_0$, let

$$\mathbb{P}_n := \begin{bmatrix} [P_{1,n}(0)]^{-*} & 0_{q \times q} \\ 0_{q \times q} & P_{1,n}(0) \end{bmatrix} \quad (7.2)$$

and let $\mathbb{Q}_n: \mathbb{C} \rightarrow \mathbb{C}^{2q \times 2q}$ be defined by

$$\mathbb{Q}_n(z) := \begin{bmatrix} 0_{q \times q} & Q_{2,n}(0) \\ -z[Q_{2,n}(0)]^{-*} & 0_{q \times q} \end{bmatrix}. \quad (7.3)$$

Obviously, the matrix \mathbb{P}_n is invertible for all $n \in \mathbb{N}_0$ and $\mathbb{Q}_n(z)$ is invertible for all $n \in \mathbb{N}_0$ and all $z \in \mathbb{C} \setminus \{0\}$. Furthermore, let

$$\mathbb{L}_k^{(\ell)} := \begin{bmatrix} I_q & \mathbf{L}_k^{(\ell)} \\ 0_{q \times q} & I_q \end{bmatrix} \quad \text{and} \quad \mathbb{M}_k^{(\ell)} := \begin{bmatrix} I_q & 0_{q \times q} \\ -z\mathbf{M}_k^{(\ell)} & I_q \end{bmatrix} \quad (7.4)$$

for all $k, \ell \in \mathbb{N}_0$ and all $z \in \mathbb{C}$, where $[(\mathbf{L}_k^{(\ell)})_{k=0}^\infty, (\mathbf{M}_k^{(\ell)})_{k=0}^\infty]$ denotes the DS-parametrization of the ℓ -th Schur transform of $(s_j)_{j=0}^\infty$.

Lemma 7.3. *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$, then*

$$\mathbb{L}_k = \mathbb{P}_m \mathbb{L}_{k-m}^{(2m)} \mathbb{P}_m^{-1} = \mathbb{Q}_m \mathbb{M}_{k-m}^{(2m+1)} \mathbb{Q}_m^{-1}$$

and

$$\mathbb{M}_k = \mathbb{P}_m \mathbb{M}_{k-m}^{(2m)} \mathbb{P}_m^{-1}, \quad \mathbb{M}_{k+1} = \mathbb{Q}_m \mathbb{L}_{k-m}^{(2m+1)} \mathbb{Q}_m^{-1}$$

for all $k \in \mathbb{N}_0$ and $m \in \mathbb{Z}_{0,k}$.

7. The resolvent matrix corresponding to a Schur transformed moment sequence

Proof. Let $k \in \mathbb{N}_0$, let $m \in \mathbb{Z}_{0,k}$, and let $z \in \mathbb{C} \setminus \{0\}$. In view of Lemma 7.2 we have then

$$\begin{aligned}
& \mathbb{P}_m \mathbb{L}_{k-m}^{(2m)} \mathbb{P}_m^{-1} \\
&= \begin{bmatrix} [P_{1,m}(0)]^{-*} & 0_{q \times q} \\ 0_{q \times q} & P_{1,m}(0) \end{bmatrix} \begin{bmatrix} I_q & \mathbf{L}_{k-m}^{(2m)} \\ 0_{q \times q} & I_q \end{bmatrix} \begin{bmatrix} [P_{1,m}(0)]^{-*} & 0_{q \times q} \\ 0_{q \times q} & P_{1,m}(0) \end{bmatrix}^{-1} \\
&= \begin{bmatrix} [P_{1,m}(0)]^{-*} & [P_{1,m}(0)]^{-*} \mathbf{L}_{k-m}^{(2m)} \\ 0_{q \times q} & P_{1,m}(0) \end{bmatrix} \begin{bmatrix} [P_{1,m}(0)]^* & 0_{q \times q} \\ 0_{q \times q} & [P_{1,m}(0)]^{-1} \end{bmatrix} \\
&= \begin{bmatrix} I_q & [P_{1,m}(0)]^{-*} \mathbf{L}_{k-m}^{(2m)} [P_{1,m}(0)]^{-1} \\ 0_{q \times q} & I_q \end{bmatrix} = \begin{bmatrix} I_q & \mathbf{L}_k \\ 0_{q \times q} & I_q \end{bmatrix} = \mathbb{L}_k
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{Q}_m(z) \mathbb{M}_{k-m}^{(2m+1)}(z) [\mathbb{Q}_m(z)]^{-1} \\
&= \begin{bmatrix} 0_{q \times q} & Q_{2,m}(0) \\ -z[Q_{2,m}(0)]^{-*} & 0_{q \times q} \end{bmatrix} \begin{bmatrix} I_q & 0_{q \times q} \\ -z\mathbf{M}_{k-m}^{(2m+1)} & I_q \end{bmatrix} \begin{bmatrix} 0_{q \times q} & Q_{2,m}(0) \\ -z[Q_{2,m}(0)]^{-*} & 0_{q \times q} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -zQ_{2,m}(0)\mathbf{M}_{k-m}^{(2m+1)} & Q_{2,m}(0) \\ -z[Q_{2,m}(0)]^{-*} & 0_{q \times q} \end{bmatrix} \begin{bmatrix} 0_{q \times q} & -z^{-1}[Q_{2,m}(0)]^* \\ [Q_{2,m}(0)]^{-1} & 0_{q \times q} \end{bmatrix} \\
&= \begin{bmatrix} I_q & Q_{2,m}(0)\mathbf{M}_{k-m}^{(2m+1)}[Q_{2,m}(0)]^* \\ 0_{q \times q} & I_q \end{bmatrix} = \begin{bmatrix} I_q & \mathbf{L}_k \\ 0_{q \times q} & I_q \end{bmatrix} = \mathbb{L}_k
\end{aligned}$$

and furthermore

$$\begin{aligned}
& \mathbb{P}_m \mathbb{M}_{k-m}^{(2m)}(z) \mathbb{P}_m^{-1} \\
&= \begin{bmatrix} [P_{1,m}(0)]^{-*} & 0_{q \times q} \\ 0_{q \times q} & P_{1,m}(0) \end{bmatrix} \begin{bmatrix} I_q & 0_{q \times q} \\ -z\mathbf{M}_{k-m}^{(2m)} & I_q \end{bmatrix} \begin{bmatrix} [P_{1,m}(0)]^{-*} & 0_{q \times q} \\ 0_{q \times q} & P_{1,m}(0) \end{bmatrix}^{-1} \\
&= \begin{bmatrix} [P_{1,m}(0)]^{-*} & 0_{q \times q} \\ -zP_{1,m}(0)\mathbf{M}_{k-m}^{(2m)} & P_{1,m}(0) \end{bmatrix} \begin{bmatrix} [P_{1,m}(0)]^* & 0_{q \times q} \\ 0_{q \times q} & [P_{1,m}(0)]^{-1} \end{bmatrix} \\
&= \begin{bmatrix} I_q & 0_{q \times q} \\ -zP_{1,m}(0)\mathbf{M}_{k-m}^{(2m)}[P_{1,m}(0)]^* & I_q \end{bmatrix} = \begin{bmatrix} I_q & 0_{q \times q} \\ -z\mathbf{M}_k & I_q \end{bmatrix} = \mathbb{M}_k(z)
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{Q}_m(z) \mathbb{L}_{k-m}^{(2m+1)} [\mathbb{Q}_m(z)]^{-1} \\
&= \begin{bmatrix} 0_{q \times q} & Q_{2,m}(0) \\ -z[Q_{2,m}(0)]^{-*} & 0_{q \times q} \end{bmatrix} \begin{bmatrix} I_q & \mathbf{L}_{k-m}^{(2m+1)} \\ 0_{q \times q} & I_q \end{bmatrix} \begin{bmatrix} 0_{q \times q} & Q_{2,m}(0) \\ -z[Q_{2,m}(0)]^{-*} & 0_{q \times q} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 0_{q \times q} & Q_{2,m}(0) \\ -z[Q_{2,m}(0)]^{-*} & -z[Q_{2,m}(0)]^{-*} \mathbf{L}_{k-m}^{(2m+1)} \end{bmatrix} \begin{bmatrix} 0_{q \times q} & -z^{-1}[Q_{2,m}(0)]^* \\ [Q_{2,m}(0)]^{-1} & 0_{q \times q} \end{bmatrix} \\
&= \begin{bmatrix} I_q & 0_{q \times q} \\ -z[Q_{2,m}(0)]^{-*} \mathbf{L}_{k-m}^{(2m+1)} [Q_{2,m}(0)]^{-1} & I_q \end{bmatrix} = \begin{bmatrix} I_q & 0_{q \times q} \\ -z\mathbf{M}_{k+1} & I_q \end{bmatrix} = \mathbb{M}_{k+1}(z). \quad \square
\end{aligned}$$

7. The resolvent matrix corresponding to a Schur transformed moment sequence

In view of Proposition 5.4 and Definition 4.6, we are able to introduce the following notations:

Notation 7.4. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ and let $\ell \in \mathbb{N}_0$. Then, we write $[(P_{1,k}^{(\ell)})_{k=0}^\infty, (Q_{1,k}^{(\ell)})_{k=0}^\infty, (P_{2,k}^{(\ell)})_{k=0}^\infty, (Q_{2,k}^{(\ell)})_{k=0}^\infty]$ for the Stieltjes quadruple associated with the ℓ -th Schur transform of $(s_j)_{j=0}^\infty$.

Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ and $\ell \in \mathbb{N}_0$. Then, the matrices $P_{1,n}^{(\ell)}(0)$ and $Q_{2,n}^{(\ell)}(0)$ are invertible. In accordance with (7.2) and (7.3), let

$$\mathbb{P}_n^{(\ell)} := \begin{bmatrix} [P_{1,n}^{(\ell)}(0)]^{-*} & 0_{q \times q} \\ 0_{q \times q} & P_{1,n}^{(\ell)}(0) \end{bmatrix} \quad (7.5)$$

for all $n \in \mathbb{N}_0$, and let $\mathbb{Q}_n^{(\ell)} : \mathbb{C} \rightarrow \mathbb{C}^{2q \times 2q}$ be defined by

$$\mathbb{Q}_n^{(\ell)}(z) := \begin{bmatrix} 0_{q \times q} & Q_{2,n}^{(\ell)}(0) \\ -z[Q_{2,n}^{(\ell)}(0)]^{-*} & 0_{q \times q} \end{bmatrix}. \quad (7.6)$$

Obviously, the matrix $\mathbb{P}_n^{(\ell)}$ is invertible for all $n \in \mathbb{N}_0$ and $\mathbb{Q}_n^{(\ell)}(z)$ is invertible for all $n \in \mathbb{N}_0$ and all $z \in \mathbb{C} \setminus \{0\}$. For all $m \in \mathbb{N}_0$, let

$$\mathbf{Q}_m := \prod_{\ell=0}^m \mathbb{Q}_0^{(\ell)}. \quad (7.7)$$

In particular, we have $\mathbf{Q}_0 = \mathbb{Q}_0^{(0)} = \mathbb{Q}_0$.

In view of Proposition 5.4, we can apply Lemma 7.3 with $m = 0$ to the ℓ -th Schur transform of a sequence $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ and obtain:

Remark 7.5. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$, then

$$\mathbb{L}_k^{(\ell)} \mathbb{Q}_0^{(\ell)} = \mathbb{Q}_0^{(\ell)} \mathbb{M}_k^{(\ell+1)} \quad \text{and} \quad \mathbb{M}_{k+1}^{(\ell)} \mathbb{Q}_0^{(\ell)} = \mathbb{Q}_0^{(\ell)} \mathbb{L}_k^{(\ell+1)}$$

for all $k, \ell \in \mathbb{N}_0$.

By repeated application of Remark 7.5 we get:

Lemma 7.6. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$, then

$$\mathbf{Q}_{2n} \mathbb{L}_k^{(2n+1)} = \mathbb{M}_{k+n+1} \mathbf{Q}_{2n}, \quad \mathbf{Q}_{2n} \mathbb{M}_k^{(2n+1)} = \mathbb{L}_{k+n} \mathbf{Q}_{2n}$$

and

$$\mathbf{Q}_{2n+1} \mathbb{L}_k^{(2n+2)} = \mathbb{L}_{k+n+1} \mathbf{Q}_{2n+1}, \quad \mathbf{Q}_{2n+1} \mathbb{M}_k^{(2n+2)} = \mathbb{M}_{k+n+1} \mathbf{Q}_{2n+1}$$

for all $k, n \in \mathbb{N}_0$.

7. The resolvent matrix corresponding to a Schur transformed moment sequence

Proof. Using Remark 7.5 and (7.7), we obtain

$$\begin{aligned}\mathbf{Q}_0 \mathbb{L}_k^{(1)} &= \mathbf{Q}_0^{(0)} \mathbb{L}_k^{(0+1)} = \mathbb{M}_{k+1}^{(0)} \mathbf{Q}_0^{(0)} = \mathbb{M}_{k+1} \mathbf{Q}_0, \\ \mathbf{Q}_0 \mathbb{M}_k^{(1)} &= \mathbf{Q}_0^{(0)} \mathbb{M}_k^{(0+1)} = \mathbb{L}_k^{(0)} \mathbf{Q}_0^{(0)} = \mathbb{L}_k \mathbf{Q}_0, \\ \mathbf{Q}_1 \mathbb{L}_k^{(2)} &= \mathbf{Q}_0^{(0)} \mathbf{Q}_0^{(1)} \mathbb{L}_k^{(1+1)} \\ &= \mathbf{Q}_0^{(0)} \mathbb{M}_{k+1}^{(1)} \mathbf{Q}_0^{(1)} = \mathbf{Q}_0^{(0)} \mathbb{M}_{k+1}^{(0+1)} \mathbf{Q}_0^{(1)} = \mathbb{L}_{k+1}^{(0)} \mathbf{Q}_0^{(0)} \mathbf{Q}_0^{(1)} = \mathbb{L}_{k+1} \mathbf{Q}_1,\end{aligned}$$

and

$$\begin{aligned}\mathbf{Q}_1 \mathbb{M}_k^{(2)} &= \mathbf{Q}_0^{(0)} \mathbf{Q}_0^{(1)} \mathbb{M}_k^{(1+1)} \\ &= \mathbf{Q}_0^{(0)} \mathbb{L}_k^{(1)} \mathbf{Q}_0^{(1)} = \mathbf{Q}_0^{(0)} \mathbb{L}_k^{(0+1)} \mathbf{Q}_0^{(1)} = \mathbb{M}_{k+1}^{(0)} \mathbf{Q}_0^{(0)} \mathbf{Q}_0^{(1)} = \mathbb{M}_{k+1} \mathbf{Q}_1.\end{aligned}$$

for all $k \in \mathbb{N}_0$. Now let $n \in \mathbb{N}$ and suppose that

$$\mathbf{Q}_{2n-1} \mathbb{L}_j^{(2n)} = \mathbb{L}_{j+n} \mathbf{Q}_{2n-1}, \quad \mathbf{Q}_{2n-1} \mathbb{M}_j^{(2n)} = \mathbb{M}_{j+n} \mathbf{Q}_{2n-1}$$

hold true for all $j \in \mathbb{N}_0$. Taking additionally into account (7.7) and Remark 7.5, we get for all $k \in \mathbb{N}_0$ then

$$\begin{aligned}\mathbf{Q}_{2n} \mathbb{L}_k^{(2n+1)} &= \mathbf{Q}_{2n-1} \mathbf{Q}_0^{(2n)} \mathbb{L}_k^{(2n+1)} = \mathbf{Q}_{2n-1} \mathbb{M}_{k+1}^{(2n)} \mathbf{Q}_0^{(2n)} \\ &= \mathbb{M}_{k+1+n} \mathbf{Q}_{2n-1} \mathbf{Q}_0^{(2n)} = \mathbb{M}_{k+n+1} \mathbf{Q}_{2n}\end{aligned}$$

and

$$\begin{aligned}\mathbf{Q}_{2n} \mathbb{M}_k^{(2n+1)} &= \mathbf{Q}_{2n-1} \mathbf{Q}_0^{(2n)} \mathbb{M}_k^{(2n+1)} = \mathbf{Q}_{2n-1} \mathbb{L}_k^{(2n)} \mathbf{Q}_0^{(2n)} \\ &= \mathbb{L}_{k+n} \mathbf{Q}_{2n-1} \mathbf{Q}_0^{(2n)} = \mathbb{L}_{k+n} \mathbf{Q}_{2n}.\end{aligned}$$

Using this and again (7.7) and Remark 7.5, we obtain for all $\ell \in \mathbb{N}_0$ furthermore

$$\begin{aligned}\mathbf{Q}_{2n+1} \mathbb{L}_\ell^{(2n+2)} &= \mathbf{Q}_{2n} \mathbf{Q}_0^{(2n+1)} \mathbb{L}_\ell^{((2n+1)+1)} \\ &= \mathbf{Q}_{2n} \mathbb{M}_{\ell+1}^{(2n+1)} \mathbf{Q}_0^{(2n+1)} = \mathbb{L}_{\ell+1+n} \mathbf{Q}_{2n} \mathbf{Q}_0^{(2n+1)} = \mathbb{L}_{\ell+n+1} \mathbf{Q}_{2n+1}\end{aligned}$$

and

$$\begin{aligned}\mathbf{Q}_{2n+1} \mathbb{M}_\ell^{(2n+2)} &= \mathbf{Q}_{2n} \mathbf{Q}_0^{(2n+1)} \mathbb{M}_\ell^{((2n+1)+1)} \\ &= \mathbf{Q}_{2n} \mathbb{L}_\ell^{(2n+1)} \mathbf{Q}_0^{(2n+1)} = \mathbb{M}_{\ell+n+1} \mathbf{Q}_{2n} \mathbf{Q}_0^{(2n+1)} = \mathbb{M}_{\ell+n+1} \mathbf{Q}_{2n+1}.\end{aligned}$$

Thus, the assertion is proved by mathematical induction. \square

Notation 7.7. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. For all $\ell \in \mathbb{N}_0$ denote by $(U_m^{(\ell)})_{m=0}^\infty$ the sequence of Dyukarev matrix polynomials associated with the ℓ -th Schur transform of $(s_j)_{j=0}^\infty$. Then, for all $k, \ell \in \mathbb{N}_0$, let $\mathbf{U}_{k,\ell} := U_{k-\ell}^{(\ell)}$.

7. The resolvent matrix corresponding to a Schur transformed moment sequence

In particular, this means $\mathbf{U}_{k,0} := U_k$ and, in view of (4.6) and (4.7), furthermore

$$\mathbf{U}_{2n,2k} := \begin{bmatrix} \alpha_{n-k}^{(2k)} & \beta_{n-k}^{(2k)} \\ \gamma_{n-k}^{(2k)} & \delta_{n-k}^{(2k)} \end{bmatrix}, \quad \mathbf{U}_{2n,2k+1} := \begin{bmatrix} \alpha_{n-k-1}^{(2k+1)} & \beta_{n-k}^{(2k+1)} \\ \gamma_{n-k-1}^{(2k+1)} & \delta_{n-k}^{(2k+1)} \end{bmatrix}, \quad (7.8)$$

and

$$\mathbf{U}_{2n+1,2k} := \begin{bmatrix} \alpha_{n-k}^{(2k)} & \beta_{n-k+1}^{(2k)} \\ \gamma_{n-k}^{(2k)} & \delta_{n-k+1}^{(2k)} \end{bmatrix}, \quad \mathbf{U}_{2n+1,2k+1} := \begin{bmatrix} \alpha_{n-k}^{(2k+1)} & \beta_{n-k}^{(2k+1)} \\ \gamma_{n-k}^{(2k+1)} & \delta_{n-k}^{(2k+1)} \end{bmatrix} \quad (7.9)$$

for all $n, k \in \mathbb{N}_0$.

Now we are able to write down the connection between the resolvent matrices $\mathbf{U}_{m,0} = U_m$ and $\mathbf{U}_{m,1} = U_{m-1}^{(1)}$ given via (7.8) and (7.9).

Proposition 7.8. *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Then $\mathbf{U}_{m,1} = \mathbb{Q}_0^{-1} \mathbb{M}_0^{-1} \mathbf{U}_{m,0} \mathbb{Q}_0$ for all $m \in \mathbb{N}$.*

Proof. According to Proposition 5.4, we have $(s_j^{[1]})_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Proposition 4.4 yields

$$\mathbf{U}_{1,0} = \mathbb{M}_0 \mathbb{L}_0 \quad \text{and} \quad \mathbf{U}_{1,1} = \mathbb{M}_0^{(1)}.$$

Hence, using Lemma 7.3, we obtain

$$\begin{aligned} \mathbb{Q}_0^{-1} \mathbb{M}_0^{-1} \mathbf{U}_{1,0} \mathbb{Q}_0 &= \mathbb{Q}_0^{-1} \mathbb{M}_0^{-1} \mathbb{M}_0 \mathbb{L}_0 \mathbb{Q}_0 \\ &= \mathbb{Q}_0^{-1} \mathbb{M}_0^{-1} \mathbb{M}_0 \mathbb{Q}_0 \mathbb{M}_0^{(1)} \mathbb{Q}_0^{-1} \mathbb{Q}_0 = \mathbb{M}_0^{(1)} = \mathbf{U}_{1,1}. \end{aligned}$$

Let $n \in \mathbb{N}$. Proposition 4.4 yields then

$$\mathbf{U}_{2n,0} = \left(\prod_{k=0}^{n-1} \mathbb{M}_k \mathbb{L}_k \right) \mathbb{M}_n, \quad \mathbf{U}_{2n+1,0} = \prod_{k=0}^n (\mathbb{M}_k \mathbb{L}_k)$$

and

$$\mathbf{U}_{2n,1} = \prod_{k=0}^{n-1} (\mathbb{M}_k^{(1)} \mathbb{L}_k^{(1)}), \quad \mathbf{U}_{2n+1,1} = \left(\prod_{k=0}^{n-1} \mathbb{M}_k^{(1)} \mathbb{L}_k^{(1)} \right) \mathbb{M}_n^{(1)}.$$

Hence, using Lemma 7.3, we obtain

$$\begin{aligned} \mathbb{Q}_0^{-1} \mathbb{M}_0^{-1} \mathbf{U}_{2n,0} \mathbb{Q}_0 &= \mathbb{Q}_0^{-1} \mathbb{M}_0^{-1} \left(\prod_{k=0}^{n-1} \mathbb{M}_k \mathbb{L}_k \right) \mathbb{M}_n \mathbb{Q}_0 = \mathbb{Q}_0^{-1} \mathbb{M}_0^{-1} \mathbb{M}_0 \left(\prod_{k=0}^{n-1} \mathbb{L}_k \mathbb{M}_{k+1} \right) \mathbb{Q}_0 \\ &= \mathbb{Q}_0^{-1} \left(\prod_{k=0}^{n-1} \mathbb{Q}_0 \mathbb{M}_k^{(1)} \mathbb{Q}_0^{-1} \mathbb{Q}_0 \mathbb{L}_k^{(1)} \mathbb{Q}_0^{-1} \right) \mathbb{Q}_0 = \prod_{k=0}^{n-1} (\mathbb{M}_k^{(1)} \mathbb{L}_k^{(1)}) = \mathbf{U}_{2n,1} \end{aligned}$$

7. The resolvent matrix corresponding to a Schur transformed moment sequence

and

$$\begin{aligned}
\mathbb{Q}_0^{-1} \mathbb{M}_0^{-1} \mathbf{U}_{2n+1,0} \mathbb{Q}_0 &= \mathbb{Q}_0^{-1} \mathbb{M}_0^{-1} \left(\prod_{k=0}^n \mathbb{M}_k \mathbb{L}_k \right) \mathbb{Q}_0 = \mathbb{Q}_0^{-1} \mathbb{M}_0^{-1} \mathbb{M}_0 \left(\prod_{k=0}^{n-1} \mathbb{L}_k \mathbb{M}_{k+1} \right) \mathbb{L}_n \mathbb{Q}_0 \\
&= \mathbb{Q}_0^{-1} \left(\prod_{k=0}^{n-1} \mathbb{Q}_0 \mathbb{M}_k^{(1)} \mathbb{Q}_0^{-1} \mathbb{Q}_0 \mathbb{L}_k^{(1)} \mathbb{Q}_0^{-1} \right) \mathbb{Q}_0 \mathbb{M}_n^{(1)} \mathbb{Q}_0^{-1} \mathbb{Q}_0 \\
&= \left(\prod_{k=0}^{n-1} \mathbb{M}_k^{(1)} \mathbb{L}_k^{(1)} \right) \mathbb{M}_n^{(1)} = \mathbf{U}_{2n+1,1}. \quad \square
\end{aligned}$$

In view of Proposition 5.4, we can apply Proposition 7.8 to the ℓ -th Schur transform of a sequence $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ and obtain:

Remark 7.9. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Then $\mathbf{U}_{m,\ell} = \mathbb{M}_0^{(\ell)} \mathbb{Q}_0^{(\ell)} \mathbf{U}_{m,\ell+1} (\mathbb{Q}_0^{(\ell)})^{-1}$ for all $m \in \mathbb{N}$ and all $\ell \in \mathbb{Z}_{0,m-1}$.

By applying Remark 7.9 twice, we obtain:

Lemma 7.10. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Then

$$\mathbf{U}_{m,0} = \mathbb{M}_0 \mathbb{L}_0 (\mathbb{Q}_0 \mathbb{Q}_0^{(1)}) \mathbf{U}_{m,2} (\mathbb{Q}_0 \mathbb{Q}_0^{(1)})^{-1}$$

for all $m \in \mathbb{Z}_{2,\infty}$.

Proof. According to Proposition 5.4 the sequences $(s_j^{[1]})_{j=0}^\infty$ and $(s_j^{[2]})_{j=0}^\infty$ both belong to $\mathcal{K}_{q,\infty}^>$. The application of Remark 7.9 to $(s_j)_{j=0}^\infty$ and $(s_j^{[1]})_{j=0}^\infty$ yields

$$\mathbf{U}_{m,0} = \mathbb{M}_0 \mathbb{Q}_0 \mathbf{U}_{m,1} \mathbb{Q}_0^{-1} \quad \text{and} \quad \mathbf{U}_{m,1} = \mathbb{M}_0^{(1)} \mathbb{Q}_0^{(1)} \mathbf{U}_{m,2} (\mathbb{Q}_0^{(1)})^{-1}.$$

Hence, $\mathbf{U}_{m,0} = \mathbb{M}_0 \mathbb{Q}_0 \mathbb{M}_0^{(1)} \mathbb{Q}_0^{(1)} \mathbf{U}_{m,2} (\mathbb{Q}_0^{(1)})^{-1} \mathbb{Q}_0^{-1}$. From Remark 7.5 with $k = \ell = 0$, we obtain furthermore $\mathbb{Q}_0 \mathbb{M}_0^{(1)} = \mathbb{L}_0 \mathbb{Q}_0$ which completes the proof. \square

Lemma 7.11. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$, then

$$\mathbf{U}_{m,0} \mathbf{Q}_m = \prod_{\ell=0}^m (\mathbb{M}_0^{(\ell)} \mathbb{Q}_0^{(\ell)})$$

for all $m \in \mathbb{N}_0$.

Proof. According to Proposition 5.4, we have $(s_j^{[\ell]})_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ for all $\ell \in \mathbb{N}_0$. Proposition 4.4 yields

$$U_0 = \mathbb{M}_0 \quad \text{and} \quad U_1 = \mathbb{M}_0 \mathbb{L}_0.$$

7. The resolvent matrix corresponding to a Schur transformed moment sequence

Hence, we obtain

$$U_0 \mathbf{Q}_0 = \mathbb{M}_0 \mathbb{Q}_0 = \mathbb{M}_0^{(0)} \mathbb{Q}_0^{(0)}$$

and, using Remark 7.5, furthermore

$$U_1 \mathbf{Q}_1 = \mathbb{M}_0 \mathbb{L}_0 \mathbb{Q}_0^{(0)} \mathbb{Q}_0^{(1)} = \mathbb{M}_0 \mathbb{Q}_0 \mathbb{M}_0^{(1)} \mathbb{Q}_0^{(1)} = (\mathbb{M}_0^{(0)} \mathbb{Q}_0^{(0)}) (\mathbb{M}_0^{(1)} \mathbb{Q}_0^{(1)}).$$

Now let $n \in \mathbb{N}$ and suppose that

$$U_{2n-1} \mathbf{Q}_{2n-1} = \prod_{\ell=0}^{\overrightarrow{2n-1}} (\mathbb{M}_0^{(\ell)} \mathbb{Q}_0^{(\ell)})$$

holds true. Proposition 4.4 yields

$$U_{2n} = U_{2n-1} \mathbb{M}_n \quad \text{and} \quad U_{2n+1} = U_{2n} \mathbb{L}_n.$$

Hence, using Lemma 7.6 with $k = 0$, we obtain

$$U_{2n} \mathbf{Q}_{2n} = U_{2n-1} \mathbb{M}_n \mathbf{Q}_{2n-1} \mathbb{Q}_0^{(2n)} = U_{2n-1} \mathbf{Q}_{2n-1} \mathbb{M}_0^{(2n)} \mathbb{Q}_0^{(2n)} = \prod_{\ell=0}^{\overrightarrow{2n}} (\mathbb{M}_0^{(\ell)} \mathbb{Q}_0^{(\ell)})$$

and with that furthermore

$$U_{2n+1} \mathbf{Q}_{2n+1} = U_{2n} \mathbb{L}_n \mathbf{Q}_{2n} \mathbb{Q}_0^{(2n+1)} = U_{2n} \mathbf{Q}_{2n} \mathbb{M}_0^{(2n+1)} \mathbb{Q}_0^{(2n+1)} = \prod_{\ell=0}^{\overrightarrow{2n+1}} (\mathbb{M}_0^{(\ell)} \mathbb{Q}_0^{(\ell)}).$$

In view of $U_m = \mathbf{U}_{m,0}$ for all $m \in \mathbb{N}_0$, the assertion is thus proved by mathematical induction. \square

The following result expresses the resolvent matrix $\mathbf{U}_{m,0} = U_m$ in terms of the resolvent matrices $\mathbf{U}_{\ell,0} = U_\ell$ and $\mathbf{U}_{m,\ell+1} = U_{m-\ell-1}^{(\ell+1)}$, which can be considered as a splitting of the original moment problem:

Theorem 7.12. *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Then $\mathbf{U}_{m,0} = \mathbf{U}_{\ell,0} \mathbf{Q}_\ell \mathbf{U}_{m,\ell+1} \mathbf{Q}_\ell^{-1}$ for all $m \in \mathbb{N}$ and $\ell \in \mathbb{Z}_{0,m-1}$.*

Proof. Let $m \in \mathbb{N}$. According to Proposition 4.4 and Remark 7.9 we have

$$\mathbf{U}_{0,0} \mathbf{Q}_0 \mathbf{U}_{m,1} \mathbf{Q}_0^{-1} = U_0^{(0)} \mathbb{Q}_0^{(0)} \mathbf{U}_{m,1} (\mathbb{Q}_0^{(0)})^{-1} = U_0 \mathbb{Q}_0 \mathbf{U}_{m,1} \mathbb{Q}_0^{-1} = \mathbb{M}_0 \mathbb{Q}_0 \mathbf{U}_{m,1} \mathbb{Q}_0^{-1} = \mathbf{U}_{m,0}.$$

Now suppose $m \geq 2$ and that

$$\mathbf{U}_{m,0} = \mathbf{U}_{\ell,0} \mathbf{Q}_\ell \mathbf{U}_{m,\ell+1} \mathbf{Q}_\ell^{-1}$$

8. Orthogonal matrix polynomials corresponding to a transformed sequence

holds true for some $\ell \in \mathbb{Z}_{0,m-2}$. Using Lemma 7.11 and Remark 7.9, we can conclude then

$$\begin{aligned} \mathbf{U}_{\ell+1,0} \mathbf{Q}_{\ell+1} \mathbf{U}_{m,\ell+2} \mathbf{Q}_{\ell+1}^{-1} &= \left[\prod_{k=0}^{\ell+1} (\mathbb{M}_0^{(k)} \mathbb{Q}_0^{(k)}) \right] \mathbf{U}_{m,\ell+2} (\mathbf{Q}_\ell \mathbb{Q}_0^{(\ell+1)})^{-1} \\ &= \left[\prod_{k=0}^{\ell} (\mathbb{M}_0^{(k)} \mathbb{Q}_0^{(k)}) \right] (\mathbb{M}_0^{(\ell+1)} \mathbb{Q}_0^{(\ell+1)}) \mathbf{U}_{m,(\ell+1)+1} (\mathbb{Q}_0^{(\ell+1)})^{-1} \mathbf{Q}_\ell^{-1} \\ &= \mathbf{U}_{\ell,0} \mathbf{Q}_\ell \mathbf{U}_{m,\ell+1} \mathbf{Q}_\ell^{-1} = \mathbf{U}_{m,0}. \end{aligned}$$

Thus, the assertion is proved by mathematical induction. \square

8. Orthogonal matrix polynomials corresponding to a Schur transformed moment sequence

From Propositions 7.8 and 4.9 we obtain the following connection between the matrix polynomials $P_{1,n}$, $Q_{1,n}$, $P_{2,n}$, and $Q_{2,n}$, and the matrix polynomials $P_{1,n}^{(1)}$, $Q_{1,n}^{(1)}$, $P_{2,n}^{(1)}$, and $Q_{2,n}^{(1)}$ given in Notations 4.5 and 7.4 and Definition 4.6 corresponding to a Stieltjes positive definite sequence and its first Schur transform, respectively:

Proposition 8.1. *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ with first Schur transform $(s_j^{[1]})_{j=0}^\infty$. Then the Stieltjes quadruple $[(P_{1,k})_{k=0}^\infty, (Q_{1,k})_{k=0}^\infty, (P_{2,k})_{k=0}^\infty, (Q_{2,k})_{k=0}^\infty]$ of $(s_j)_{j=0}^\infty$ and $[(P_{1,k}^{(1)})_{k=0}^\infty, (Q_{1,k}^{(1)})_{k=0}^\infty, (P_{2,k}^{(1)})_{k=0}^\infty, (Q_{2,k}^{(1)})_{k=0}^\infty]$ of $(s_j^{[1]})_{j=0}^\infty$, resp., fulfill*

$$\begin{aligned} [P_{1,n}^{(1)}(z)] [P_{1,n}^{(1)}(0)]^{-1} &= [s_0^{-1} Q_{2,n}(z)] [s_0^{-1} Q_{2,n}(0)]^{-1}, \\ [Q_{1,n}^{(1)}(z)] [P_{1,n}^{(1)}(0)]^{-1} &= [Q_{2,n}(z) - s_0 P_{2,n}(z)] [s_0^{-1} Q_{2,n}(0)]^{-1}, \\ [P_{2,n-1}^{(1)}(z)] [Q_{2,n-1}^{(1)}(0)]^{-1} &= [s_0^{-1} Q_{1,n}(z)] [-s_0 P_{1,n}(0)]^{-1}, \end{aligned}$$

and

$$[Q_{2,n-1}^{(1)}(z)] [Q_{2,n-1}^{(1)}(0)]^{-1} = [z Q_{1,n}(z) - s_0 P_{1,n}(z)] [-s_0 P_{1,n}(0)]^{-1}.$$

for all $n \in \mathbb{N}$ and all $z \in \mathbb{C}$.

Proof. According to Proposition 5.4, we have $(s_j^{[1]})_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. In view of (4.8), (7.3), and $s_0^* = s_0$, we get

$$\mathbb{M}_0(z) = \begin{bmatrix} I_q & 0_{q \times q} \\ -zs_0^{-1} & I_q \end{bmatrix} \quad \text{and} \quad \mathbb{Q}_0(z) = \begin{bmatrix} 0_{q \times q} & s_0 \\ -zs_0^{-1} & 0_{q \times q} \end{bmatrix}.$$

Hence,

$$\mathbb{M}_0(z) \mathbb{Q}_0(z) = \begin{bmatrix} 0_{q \times q} & s_0 \\ -zs_0^{-1} & -zI_q \end{bmatrix}$$

8. Orthogonal matrix polynomials corresponding to a transformed sequence

and thus

$$[\mathbb{Q}_0(z)]^{-1}[\mathbb{M}_0(z)]^{-1} = [\mathbb{M}_0(z)\mathbb{Q}_0(z)]^{-1} = \begin{bmatrix} -I_q & -z^{-1}s_0 \\ s_0^{-1} & 0_{q \times q} \end{bmatrix}.$$

Using (7.8) and Proposition 7.8 we obtain then

$$\begin{aligned} \begin{bmatrix} \alpha_{n-1}^{(1)}(z) & \beta_n^{(1)}(z) \\ \gamma_{n-1}^{(1)}(z) & \delta_n^{(1)}(z) \end{bmatrix} &= \mathbf{U}_{2n,1}(z) = [\mathbb{Q}_0(z)]^{-1}[\mathbb{M}_0(z)]^{-1}\mathbf{U}_{2n,0}(z)\mathbb{Q}_0(z) \\ &= \begin{bmatrix} -I_q & -z^{-1}s_0 \\ s_0^{-1} & 0_{q \times q} \end{bmatrix} \begin{bmatrix} \alpha_n^{(0)}(z) & \beta_n^{(0)}(z) \\ \gamma_n^{(0)}(z) & \delta_n^{(0)}(z) \end{bmatrix} \begin{bmatrix} 0_{q \times q} & s_0 \\ -zs_0^{-1} & 0_{q \times q} \end{bmatrix} \\ &= \begin{bmatrix} [z\beta_n^{(0)}(z) + s_0\delta_n^{(0)}(z)]s_0^{-1} & -[\alpha_n^{(0)}(z) + z^{-1}s_0\gamma_n^{(0)}(z)]s_0 \\ -zs_0^{-1}\beta_n^{(0)}(z)s_0^{-1} & s_0^{-1}\alpha_n^{(0)}(z)s_0 \end{bmatrix}. \end{aligned}$$

Since (7.8) and Proposition 4.9 yield

$$\begin{bmatrix} \alpha_n^{(0)}(z) & \beta_n^{(0)}(z) \\ \gamma_n^{(0)}(z) & \delta_n^{(0)}(z) \end{bmatrix} = \begin{bmatrix} [Q_{2,n}(z)][Q_{2,n}(0)]^{-1} & -[Q_{1,n}(z)][P_{1,n}(0)]^{-1} \\ -z[P_{2,n}(z)][Q_{2,n}(0)]^{-1} & [P_{1,n}(z)][P_{1,n}(0)]^{-1} \end{bmatrix}$$

and

$$\begin{bmatrix} \alpha_{n-1}^{(1)}(z) & \beta_n^{(1)}(z) \\ \gamma_{n-1}^{(1)}(z) & \delta_n^{(1)}(z) \end{bmatrix} = \begin{bmatrix} [Q_{2,n-1}^{(1)}(z)][Q_{2,n-1}^{(1)}(0)]^{-1} & -[Q_{1,n}^{(1)}(z)][P_{1,n}^{(1)}(0)]^{-1} \\ -z[P_{2,n-1}^{(1)}(z)][Q_{2,n-1}^{(1)}(0)]^{-1} & [P_{1,n}^{(1)}(z)][P_{1,n}^{(1)}(0)]^{-1} \end{bmatrix}$$

we can conclude

$$\begin{aligned} [Q_{2,n-1}^{(1)}(z)][Q_{2,n-1}^{(1)}(0)]^{-1} &= [z(-[Q_{1,n}(z)][P_{1,n}(0)]^{-1}) + s_0[P_{1,n}(z)][P_{1,n}(0)]^{-1}]s_0^{-1}, \\ -[Q_{1,n}^{(1)}(z)][P_{1,n}^{(1)}(0)]^{-1} &= -[Q_{2,n}(z)][Q_{2,n}(0)]^{-1} + z^{-1}s_0(-z[P_{2,n}(z)][Q_{2,n}(0)]^{-1})s_0, \\ -z[P_{2,n-1}^{(1)}(z)][Q_{2,n-1}^{(1)}(0)]^{-1} &= -zs_0^{-1}(-[Q_{1,n}(z)][P_{1,n}(0)]^{-1})s_0^{-1}, \end{aligned}$$

and

$$[P_{1,n}^{(1)}(z)][P_{1,n}^{(1)}(0)]^{-1} = s_0^{-1}([Q_{2,n}(z)][Q_{2,n}(0)]^{-1})s_0.$$

Hence, the asserted identities follow. \square

Lemma 8.2. *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Then*

$$P_{1,n}^{(1)}(0) = s_0^{-1}Q_{2,n}(0) \quad \text{and} \quad Q_{2,n}^{(1)}(0) = -s_0P_{1,n+1}(0)$$

for all $n \in \mathbb{N}$.

Proof. According to Proposition 5.4, we have $(s_j^{[1]})_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Denote by $[(\mathbf{L}_k)_{k=0}^\infty, (\mathbf{M}_k)_{k=0}^\infty]$ and $[(\mathbf{L}_k^{(1)})_{k=0}^\infty, (\mathbf{M}_k^{(1)})_{k=0}^\infty]$ the DS-parametrization of $(s_j)_{j=0}^\infty$ and

8. Orthogonal matrix polynomials corresponding to a transformed sequence

$(s_j^{[1]})_{j=0}^\infty$, respectively. Let $n \in \mathbb{N}$. Using Remarks 4.8 and 6.6 we have, in view of (3.2), then

$$\begin{aligned} P_{1,n}^{(1)}(0) &= (-1)^n \left(\prod_{k=0}^{n-1} \mathbf{M}_k^{(1)} \mathbf{L}_k^{(1)} \right)^{-1} = (-1)^n \left(\prod_{k=0}^{n-1} s_0^{-1} \mathbf{L}_k s_0^{-1} s_0 \mathbf{M}_{k+1} s_0 \right)^{-1} \\ &= (-1)^n \left[s_0^{-1} \mathbf{L}_0 \left(\prod_{\ell=1}^{n-1} \mathbf{M}_\ell \mathbf{L}_\ell \right) \mathbf{M}_n s_0 \right]^{-1} = (-1)^n \left[\mathbf{M}_0 \mathbf{L}_0 \left(\prod_{\ell=1}^{n-1} \mathbf{M}_\ell \mathbf{L}_\ell \right) \mathbf{M}_n s_0 \right]^{-1} \\ &= s_0^{-1} \left((-1)^n \left[\left(\prod_{\ell=0}^{n-1} \mathbf{M}_\ell \mathbf{L}_\ell \right) \mathbf{M}_n \right]^{-1} \right) = s_0^{-1} Q_{2,n}(0) \end{aligned}$$

and

$$\begin{aligned} Q_{2,n}^{(1)}(0) &= (-1)^n \left[\left(\prod_{k=0}^{n-1} \mathbf{M}_k^{(1)} \mathbf{L}_k^{(1)} \right) \mathbf{M}_n^{(1)} \right]^{-1} \\ &= (-1)^n \left[\left(\prod_{k=0}^{n-1} s_0^{-1} \mathbf{L}_k s_0^{-1} s_0 \mathbf{M}_{k+1} s_0 \right) s_0^{-1} \mathbf{L}_n s_0^{-1} \right]^{-1} \\ &= (-1)^n \left[s_0^{-1} \mathbf{L}_0 \left(\prod_{\ell=1}^n \mathbf{M}_\ell \mathbf{L}_\ell \right) s_0^{-1} \right]^{-1} = (-1)^n \left[\mathbf{M}_0 \mathbf{L}_0 \left(\prod_{\ell=1}^n \mathbf{M}_\ell \mathbf{L}_\ell \right) s_0^{-1} \right]^{-1} \\ &= -s_0 \left[(-1)^{n+1} \left(\prod_{\ell=0}^n \mathbf{M}_\ell \mathbf{L}_\ell \right)^{-1} \right] = -s_0 P_{1,n+1}(0). \quad \square \end{aligned}$$

Theorem 8.3. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ with first Schur transform $(s_j^{[1]})_{j=0}^\infty$. Then the Stieltjes quadruple $[(P_{1,k})_{k=0}^\infty, (Q_{1,k})_{k=0}^\infty, (P_{2,k})_{k=0}^\infty, (Q_{2,k})_{k=0}^\infty]$ of $(s_j)_{j=0}^\infty$ and $[(P_{1,k}^{(1)})_{k=0}^\infty, (Q_{1,k}^{(1)})_{k=0}^\infty, (P_{2,k}^{(1)})_{k=0}^\infty, (Q_{2,k}^{(1)})_{k=0}^\infty]$ of $(s_j^{[1]})_{j=0}^\infty$, resp., fulfill

$$\begin{aligned} P_{1,n}^{(1)}(z) &= s_0^{-1} Q_{2,n}(z), & Q_{1,n}^{(1)}(z) &= Q_{2,n}(z) - s_0 P_{2,n}(z), \\ P_{2,n-1}^{(1)}(z) &= s_0^{-1} Q_{1,n}(z), & Q_{2,n-1}^{(1)}(z) &= z Q_{1,n}(z) - s_0 P_{1,n}(z) \end{aligned}$$

for all $n \in \mathbb{N}$ and all $z \in \mathbb{C}$.

Proof. Combine Proposition 8.1 with Lemma 8.2. \square

Proposition 8.4. Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ with Stieltjes quadruple $[(P_{1,k})_{k=0}^\infty, (Q_{1,k})_{k=0}^\infty, (P_{2,k})_{k=0}^\infty, (Q_{2,k})_{k=0}^\infty]$ and first Schur transform $(s_j^{[1]})_{j=0}^\infty$. Then

8. Orthogonal matrix polynomials corresponding to a transformed sequence

the sequences $(s_j^{[1]})_{j=0}^\infty$ and $(s_{j+1}^{[1]})_{j=0}^\infty$ both belong to $\mathcal{K}_{q,\infty}^>$. Furthermore $(s_0^{-1}Q_{2,k})_{k=0}^\infty$ is a monic right orthogonal system of matrix polynomials with respect to $(s_j^{[1]})_{j=0}^\infty$ and $(s_0^{-1}Q_{1,k+1})_{k=0}^\infty$ is a monic right orthogonal system of matrix polynomials with respect to $(s_{j+1}^{[1]})_{j=0}^\infty$. In particular, if $\mu_1 \in \mathcal{M}_\geq^q[[0, \infty); (s_j^{[1]})_{j=0}^\infty, =]$, we have the orthogonality relations

$$\int_{[0, \infty)} \left[s_0^{-1}Q_{2,m}(t) \right]^* \mu_1(dt) \left[s_0^{-1}Q_{2,n}(t) \right] = \begin{cases} 0_{q \times q}, & \text{if } m \neq n \\ L_{2,n}, & \text{if } m = n \end{cases}$$

and

$$\int_{[0, \infty)} \left[s_0^{-1}Q_{1,m}(t) \right]^* \mu_2(dt) \left[s_0^{-1}Q_{1,n}(t) \right] = \begin{cases} 0_{q \times q}, & \text{if } m \neq n \\ L_{1,n}, & \text{if } m = n \end{cases}$$

for all $m, n \in \mathbb{N}_0$, where $\mu_2: \mathfrak{B}_{[0, \infty)} \rightarrow \mathbb{C}_{\geq}^{q \times q}$ is defined by $\mu_2(B) := \int_{[0, \infty)} t \mu_1(dt)$ and belongs to $\mathcal{M}_\geq^q[[0, \infty); (s_{j+1}^{[1]})_{j=0}^\infty, =]$.

Proof. In view of Proposition 5.4 and Remark 2.5, we see that $(s_j^{[1]})_{j=0}^\infty$ and $(s_{j+1}^{[1]})_{j=0}^\infty$ both belong to $\mathcal{K}_{q,\infty}^>$. Since $Q_{2,0}(z) = s_0$ for all $z \in \mathbb{C}$, the combination of Theorem 8.3 with Remarks 4.7 and 6.1 completes the proof. \square

Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^>$. Then we want to draw the attention to two distinguished elements of the solution set $\mathcal{M}_\geq^q[[0, \infty); (s_j)_{j=0}^m, \leq]$. This concerns those measures $\underline{\sigma}_m$ and $\overline{\sigma}_m$, respectively, the Stieltjes transforms of which are generated via Theorem 4.3(a) by the two constant pairs $(\iota, \theta), (\theta, \iota) \in \hat{\mathcal{S}}_q$, where ι and θ are the constant functions in $\mathbb{C} \setminus [0, \infty)$ with values I_q and $0_{q \times q}$, respectively. The measures $\underline{\sigma}_m$ and $\overline{\sigma}_m$ are called the *lower* and *upper extremal elements* of $\mathcal{M}_\geq^q[[0, \infty); (s_j)_{j=0}^m, \leq]$, respectively. In view of Theorem 4.3, (4.6), (4.7), and Proposition 4.9, for $n \in \mathbb{N}_0$ and $z \in \mathbb{C} \setminus [0, \infty)$ we infer for the corresponding Stieltjes transforms

$$S_{\underline{\sigma}_{2n}}(z) = [\alpha_n(z)][\gamma_n(z)]^{-1} = -[Q_{2,n}(z)][zP_{2,n}(z)]^{-1}, \quad (8.1)$$

$$S_{\overline{\sigma}_{2n}}(z) = [\beta_n(z)][\delta_n(z)]^{-1} = -[Q_{1,n}(z)][P_{1,n}(z)]^{-1} \quad (8.2)$$

and

$$S_{\underline{\sigma}_{2n+1}}(z) = [\alpha_n(z)][\gamma_n(z)]^{-1} = -[Q_{2,n}(z)][zP_{2,n}(z)]^{-1}, \quad (8.3)$$

$$S_{\overline{\sigma}_{2n+1}}(z) = [\beta_{n+1}(z)][\delta_{n+1}(z)]^{-1} = -[Q_{1,n+1}(z)][P_{1,n+1}(z)]^{-1}. \quad (8.4)$$

The functions introduced in (8.1)–(8.4) play an important role in the considerations of Yu. M. Dyukarev [19]. We refer the reader to [19, Section 3] for a detailed discussion of these functions and their extremality properties.

Taking into account Notation 4.5, we see that $S_{\underline{\sigma}_{2n+1}}$ coincides with $S_{\underline{\sigma}_{2n}}$ and does not depend on s_{2n+1} and that $S_{\overline{\sigma}_{2n}}$ coincides with $S_{\overline{\sigma}_{2n-1}}$ and does not depend on s_{2n} . In particular

$$\underline{\sigma}_{2n+1} = \underline{\sigma}_{2n} \quad \text{and} \quad \overline{\sigma}_{2n} = \overline{\sigma}_{2n-1}. \quad (8.5)$$

8. Orthogonal matrix polynomials corresponding to a transformed sequence

Since $\underline{\sigma}_m$ and $\bar{\sigma}_m$ both belong to $\mathcal{M}_{\geq}^q[[0, \infty); (s_j)_{j=0}^m, \leq]$, we can hence conclude

$$\underline{\sigma}_{2n} \in \mathcal{M}_{\geq}^q[[0, \infty); (s_j)_{j=0}^{2n}, =], \quad \bar{\sigma}_{2n-1} \in \mathcal{M}_{\geq}^q[[0, \infty); (s_j)_{j=0}^{2n-1}, =]. \quad (8.6)$$

The lower and upper extremal elements $\underline{\sigma}_m$ and $\bar{\sigma}_m$ of $\mathcal{M}_{\geq}^q[[0, \infty); (s_j)_{j=0}^m, \leq]$ are concentrated on a finite number of points in $[0, \infty)$. In particular, they possess power moments up to any order, which coincide with zero Stieltjes parameter extensions introduced in Definition 2.9:

Lemma 8.5. *Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n-1} \in \mathcal{K}_{q,2n-1}^>$. Then $\int_{[0,\infty)} x^j \bar{\sigma}_{2n-1}(dx) = s_j^\circ$ for all $j \in \mathbb{N}_0$, where $(s_j^\circ)_{j=0}^\infty$ is the zero Stieltjes parameter extension of $(s_j)_{j=0}^{2n-1}$.*

Proof. For all $j \in \mathbb{N}_0$ let $t_j := \int_{[0,\infty)} x^j \bar{\sigma}_{2n-1}(dx)$. According to Theorem 1.1, we have then $(t_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^\geq$. Because of (8.6), we get furthermore $s_j = t_j$ for all $j \in \mathbb{Z}_{0,2n-1}$. From Proposition 2.2 and (2.2) we can consequently conclude that the matrix $t_{2n} - \Theta_{2n}$ is non-negative Hermitian, where $\Theta_{2n} := z_{n,2n-1} H_{1,n-1}^{-1} y_{n,2n-1}$. Now, we consider an arbitrary $\epsilon > 0$. Let $s_{2n} := \Theta_{2n} + \epsilon I_q$ and denote by $(\mathfrak{s}_j)_{j=0}^{2n}$ the Stieltjes parametrization of $(s_j)_{j=0}^{2n}$. In view of (2.2), the matrix $\mathfrak{t}_{2n} = L_{1,n} = \epsilon I_q$ is positive Hermitian. From Proposition 2.6 we then can easily conclude $(s_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n}^>$. As an element of $\mathcal{M}_{\geq}^q[[0, \infty); (s_j)_{j=0}^{2n}, \leq]$, the measure $\bar{\sigma}_{2n}$ fulfills $s_{2n} - \int_{[0,\infty)} x^{2n} \bar{\sigma}_{2n}(dx) \in \mathbb{C}_{\geq}^{q \times q}$. Using (8.5), we obtain

$$\Theta_{2n} \leq t_{2n} = \int_{[0,\infty)} x^{2n} \bar{\sigma}_{2n}(dx) \leq s_{2n} = \Theta_{2n} + \epsilon I_q.$$

Since this holds true for all $\epsilon > 0$, we get $t_{2n} = \Theta_{2n}$. Thus, the sequence $(t_j)_{j=0}^\infty$ belongs to $\mathcal{K}_{q,\infty}^{\geq, \text{cd}, 2n}$. Hence, we can apply Lemma 2.11 to see that $(t_j)_{j=0}^\infty$ is the zero Stieltjes parameter extension of $(t_j)_{j=0}^{2n-1}$. Because of $s_j = t_j$ for all $j \in \mathbb{Z}_{0,2n-1}$, then $t_j = s_j^\circ$ for all $j \in \mathbb{N}_0$, which completes the proof. \square

Lemma 8.6. *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n}^>$. Then $\int_{[0,\infty)} x^j \underline{\sigma}_{2n}(dx) = s_j^\circ$ for all $j \in \mathbb{N}_0$, where $(s_j^\circ)_{j=0}^\infty$ is the zero Stieltjes parameter extension of $(s_j)_{j=0}^{2n}$.*

Proof. For all $j \in \mathbb{N}_0$ let $t_j := \int_{[0,\infty)} x^j \underline{\sigma}_{2n}(dx)$. According to Theorem 1.1, we have then $(t_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^\geq$. Because of (8.6), we get furthermore $s_j = t_j$ for all $j \in \mathbb{Z}_{0,2n}$. From Proposition 2.2 and (2.3) we can consequently conclude that the matrix $t_{2n+1} - \Theta_{2n+1}$ is non-negative Hermitian, where $\Theta_1 := 0_{q \times q}$ and $\Theta_{2n+1} := z_{n+1,2n} H_{2,n-1}^{-1} y_{n+1,2n}$ for $n \in \mathbb{N}$. Now, we consider an arbitrary $\epsilon > 0$. Let $s_{2n+1} := \Theta_{2n+1} + \epsilon I_q$ and denote by $(\mathfrak{s}_j)_{j=0}^{2n+1}$ the Stieltjes parametrization of $(s_j)_{j=0}^{2n+1}$. In view of (2.3), the matrix $\mathfrak{t}_{2n+1} = L_{2,n} = \epsilon I_q$ is positive Hermitian. From Proposition 2.6 we then can easily conclude $(s_j)_{j=0}^{2n+1} \in \mathcal{K}_{q,2n+1}^>$. As an element of $\mathcal{M}_{\geq}^q[[0, \infty); (s_j)_{j=0}^{2n+1}, \leq]$, the measure $\underline{\sigma}_{2n+1}$ fulfills $s_{2n+1} - \int_{[0,\infty)} x^{2n+1} \underline{\sigma}_{2n+1}(dx) \in \mathbb{C}_{\geq}^{q \times q}$. Using (8.5), we obtain

$$\Theta_{2n+1} \leq t_{2n+1} = \int_{[0,\infty)} x^{2n+1} \underline{\sigma}_{2n+1}(dx) \leq s_{2n+1} = \Theta_{2n+1} + \epsilon I_q.$$

8. Orthogonal matrix polynomials corresponding to a transformed sequence

Since this holds true for all $\epsilon > 0$, we get $t_{2n+1} = \Theta_{2n+1}$. Thus, the sequence $(t_j)_{j=0}^\infty$ belongs to $\mathcal{K}_{q,\infty}^{\geq, \text{cd}, 2n+1}$. Hence, we can apply Lemma 2.11 to see that $(t_j)_{j=0}^\infty$ is the zero Stieltjes parameter extension of $(t_j)_{j=0}^{2n}$. Because of $s_j = t_j$ for all $j \in \mathbb{Z}_{0,2n}$, then $t_j = s_j^\circ$ for all $j \in \mathbb{N}_0$, which completes the proof. \square

In combination with (8.1)–(8.4), Theorem 8.3 yields a relation between the lower and upper extremal elements associated with a Stieltjes positive definite sequence and its first Schur transform:

Proposition 8.7. *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ with first Schur transform $(s_j^{[1]})_{j=0}^\infty$. Then $(s_j)_{j=0}^m$ and $(s_j^{[1]})_{j=0}^m$ both belong to $\mathcal{K}_{q,m}^>$ for all $m \in \mathbb{N}_0$. For all $m \in \mathbb{N}_0$ denote by $\underline{\sigma}_m$ and $\bar{\sigma}_m$ the lower and upper extremal element of $\mathcal{M}_{\geq}^q[[0, \infty); (s_j)_{j=0}^m, \leq]$. Furthermore, for all $m \in \mathbb{N}$, let $\underline{\sigma}_m^{(1)}$ and $\bar{\sigma}_m^{(1)}$ be the lower and upper extremal element of $\mathcal{M}_{\geq}^q[[0, \infty); (s_j^{[1]})_{j=0}^m, \leq]$. Then*

$$S_{\underline{\sigma}_{2n-2}^{(1)}}(z) = S_{\underline{\sigma}_{2n-1}^{(1)}}(z) = -s_0 - s_0[zS_{\bar{\sigma}_{2n}}(z)]^{-1}s_0 \quad (8.7)$$

and

$$S_{\bar{\sigma}_{2n-1}^{(1)}}(z) = S_{\bar{\sigma}_{2n}^{(1)}}(z) = -s_0 - s_0[zS_{\underline{\sigma}_{2n}}(z)]^{-1}s_0 \quad (8.8)$$

for all $n \in \mathbb{N}$ and all $z \in \mathbb{C} \setminus [0, \infty)$.

Proof. According to Proposition 5.4, we have $(s_j^{[1]})_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$. Denote by $[(P_{1,k})_{k=0}^\infty, (Q_{1,k})_{k=0}^\infty, (P_{2,k})_{k=0}^\infty, (Q_{2,k})_{k=0}^\infty]$ the Stieltjes quadruple of $(s_j)_{j=0}^\infty$ and by $[(P_{1,k}^{(1)})_{k=0}^\infty, (Q_{1,k}^{(1)})_{k=0}^\infty, (P_{2,k}^{(1)})_{k=0}^\infty, (Q_{2,k}^{(1)})_{k=0}^\infty]$ the Stieltjes quadruple of $(s_j^{[1]})_{j=0}^\infty$. Let $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus [0, \infty)$. In view of (8.1) and (8.3), we have

$$S_{\underline{\sigma}_{2n-2}^{(1)}}(z) = S_{\underline{\sigma}_{2n-1}^{(1)}}(z) = -[Q_{2,n-1}^{(1)}(z)][zP_{2,n-1}^{(1)}(z)]^{-1}.$$

Using Theorem 8.3 we obtain furthermore

$$\begin{aligned} -[Q_{2,n-1}^{(1)}(z)][zP_{2,n-1}^{(1)}(z)]^{-1} &= -[zQ_{1,n}(z) - s_0P_{1,n}(z)][zs_0^{-1}Q_{1,n}(z)]^{-1} \\ &= -s_0 + \frac{1}{z}s_0[P_{1,n}(z)][Q_{1,n}(z)]^{-1}s_0. \end{aligned}$$

Taking additionally into account (8.2), (8.7) follows.

In view of (8.2) and (8.4), we have

$$S_{\bar{\sigma}_{2n-1}^{(1)}}(z) = S_{\bar{\sigma}_{2n}^{(1)}}(z) = -[Q_{1,n}^{(1)}(z)][P_{1,n}^{(1)}(z)]^{-1}.$$

Using Theorem 8.3 we obtain furthermore

$$\begin{aligned} -[Q_{1,n}^{(1)}(z)][P_{1,n}^{(1)}(z)]^{-1} &= -[Q_{2,n}(z) - s_0P_{2,n}(z)][s_0^{-1}Q_{2,n}(z)]^{-1} \\ &= -s_0 + s_0[P_{2,n}(z)][Q_{2,n}(z)]^{-1}s_0. \end{aligned}$$

Taking additionally into account (8.1), (8.8) follows. \square

8. Orthogonal matrix polynomials corresponding to a transformed sequence

From the identities derived in the proof of Proposition 8.7, we can easily obtain the following relations between the matrix polynomials $[(P_{1,k})_{k=0}^\infty, (Q_{1,k})_{k=0}^\infty, (P_{2,k})_{k=0}^\infty, (Q_{2,k})_{k=0}^\infty]$ and $[(P_{1,k}^{(1)})_{k=0}^\infty, (Q_{1,k}^{(1)})_{k=0}^\infty, (P_{2,k}^{(1)})_{k=0}^\infty, (Q_{2,k}^{(1)})_{k=0}^\infty]$:

Theorem 8.8. *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ with first Schur transform $(s_j^{[1]})_{j=0}^\infty$ and Stieltjes quadruple $[(P_{1,k})_{k=0}^\infty, (Q_{1,k})_{k=0}^\infty, (P_{2,k})_{k=0}^\infty, (Q_{2,k})_{k=0}^\infty]$. Then $(s_j^{[1]})_{j=0}^\infty$ belongs to $\mathcal{K}_{q,\infty}^>$. Denote by $[(P_{1,k}^{(1)})_{k=0}^\infty, (Q_{1,k}^{(1)})_{k=0}^\infty, (P_{2,k}^{(1)})_{k=0}^\infty, (Q_{2,k}^{(1)})_{k=0}^\infty]$ the Stieltjes quadruple associated with $(s_j^{[1]})_{j=0}^\infty$. Then*

$$s_0[P_{1,n}(z)][Q_{1,n}(z)]^{-1}s_0 + [Q_{2,n-1}^{(1)}(z)][P_{2,n-1}^{(1)}(z)]^{-1} = zs_0$$

and

$$s_0[P_{2,n}(z)][Q_{2,n}(z)]^{-1}s_0 + [Q_{1,n}^{(1)}(z)][P_{1,n}^{(1)}(z)]^{-1} = s_0$$

for all $n \in \mathbb{N}$ and all $z \in \mathbb{C} \setminus [0, \infty)$.

Note that similar interrelations as exposed in Theorems 8.3 and 8.8 between the polynomials $[(P_{1,k})_{k=0}^\infty, (Q_{1,k})_{k=0}^\infty, (P_{2,k})_{k=0}^\infty, (Q_{2,k})_{k=0}^\infty]$ and $[(P_{1,k}^{(1)})_{k=0}^\infty, (Q_{1,k}^{(1)})_{k=0}^\infty, (P_{2,k}^{(1)})_{k=0}^\infty, (Q_{2,k}^{(1)})_{k=0}^\infty]$ were in the scalar case considered in [14]. Proposition 8.7 can also be seen from the following matrix continued fraction expansions, which appear in connection with matrix Hurwitz type polynomials in [13]. For $A, B \in \mathbb{C}^{q \times q}$ with B invertible, set $\frac{A}{B} := AB^{-1}$.

Proposition 8.9 ([12, Theorem 3.4]). *Let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ with DS-parametrization $[(\mathbf{L}_k)_{k=0}^\infty, (\mathbf{M}_k)_{k=0}^\infty]$. For all $n \in \mathbb{N}_0$ and all $z \in \mathbb{C} \setminus [0, \infty)$, then*

$$S_{\mathcal{L}_{2n}}(z) = \frac{I_q}{-z\mathbf{M}_0 + \frac{I_q}{\mathbf{L}_0 + \frac{I_q}{\ddots + \frac{I_q}{-z\mathbf{M}_{n-1} + \frac{I_q}{\mathbf{L}_{n-1} - z^{-1}\mathbf{M}_n^{-1}}}}}$$

and

$$S_{\overline{\mathcal{L}}_{2n}}(z) = \frac{I_q}{-z\mathbf{M}_0 + \frac{I_q}{\mathbf{L}_0 + \frac{I_q}{\ddots + \frac{I_q}{\mathbf{L}_{n-2} + \frac{I_q}{-z\mathbf{M}_{n-1} + \mathbf{L}_{n-1}^{-1}}}}}$$

A. Orthogonal matrix polynomials on $[0, \infty)$

Let us recall some notions on orthogonal matrix polynomials (OMP) which were used in [11, 15]. Let P be a complex $p \times q$ matrix polynomial. For all $n \in \mathbb{N}_0$, let

$$Y_n^{[P]} := \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{bmatrix},$$

where $(A_j)_{j=0}^\infty$ is the unique sequence of complex $p \times q$ matrices such that for all $z \in \mathbb{C}$ the polynomial P admits the representation $P(z) = \sum_{j=0}^\infty z^j A_j$. Furthermore, we denote by $\deg P := \sup\{j \in \mathbb{N}_0 \mid A_j \neq 0_{p \times q}\}$ the *degree* of P . Observe that in the case $P(z) = 0_{p \times q}$ for all $z \in \mathbb{C}$ we have thus $\deg P = -\infty$. If $k := \deg P \geq 0$, we refer to A_k as the *leading coefficient* of P .

Definition A.1. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{2\kappa}$ be a sequence of complex $q \times q$ matrices. A sequence $(P_k)_{k=0}^\kappa$ of complex $q \times q$ matrix polynomials is called a *monic right orthogonal system of matrix polynomials with respect to $(s_j)_{j=0}^{2\kappa}$* if the following three conditions are fulfilled:

- (I) $\deg P_k = k$ for all $k \in \mathbb{Z}_{0,\kappa}$.
- (II) P_k has the leading coefficient I_q for all $k \in \mathbb{Z}_{0,\kappa}$.
- (III) $(Y_n^{[P_j]})^* H_{1,n} Y_n^{[P_k]} = 0_{q \times q}$ for all $j, k \in \mathbb{Z}_{0,\kappa}$ with $j \neq k$, where $n := \max\{j, k\}$.

Remark A.2 (cf. [15, Remark 3.6, p. 1652]). Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{2\kappa}$ be a sequence of complex $q \times q$ matrices such that the block Hankel matrix $H_{1,n}$ is positive Hermitian for all $n \in \mathbb{Z}_{0,\kappa}$. Denote by $(P_k)_{k=0}^\kappa$ the monic right orthogonal system of matrix polynomials with respect to $(s_j)_{j=0}^{2\kappa}$. Let σ be a non-negative Hermitian $q \times q$ measure on a non-empty Borel subset Ω of \mathbb{R} satisfying $s_j = \int_\Omega t^j \sigma(dt)$ for all $j \in \mathbb{Z}_{0,2\kappa}$. Then

$$\int_\Omega [P_j(t)]^* \sigma(dt) [P_k(t)] = \begin{cases} 0_{q \times q}, & \text{if } j \neq k \\ L_{1,n}, & \text{if } j = k \end{cases}$$

for all $j, k \in \mathbb{Z}_{0,\kappa}$.

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